Geometric GRT actions and Deligne's conjecture on Hochschild cochains

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The Setup

The absolute Galois group and GRT

Grothendieck '84: Study $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via its actions on profinite fundamental group of moduli spaces of marked curves.

Ihara '90: This yields inclusion $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \hat{\mathrm{GT}}$ and $\hat{\mathrm{GT}} \subseteq \hat{\mathbb{Z}}^{\times} \times \hat{\mathbb{F}}_2$ can be given by explicit equations.

Drinfeld '90: Pro-unipotent version $\mathrm{GT}(k) \subseteq k^{\times} \times \hat{\mathbb{F}}_2(k)$ acting on quasitriangular quasi-Hopf algebras over $k[[\hbar]]$.

Brown '14: We have an inclusion $\mathcal{G}_{\mathrm{dR}}^{\mathrm{mot}} \hookrightarrow \mathrm{GT}(k)$.

Goal

We want to study $\mathcal{G}_{\mathrm{dR}}^{\mathrm{mot}}$ through understanding $\mathrm{GT}(k)$.

GRT as homotopy operad automorphism group

B. Fresse:

$$\mathrm{GT}(k) \circlearrowright \widehat{\mathcal{P}a\mathcal{B}}(k) \xrightarrow{\ \cong\ } \mathrm{Grp}(\widehat{\mathcal{P}a\mathcal{CD}}) \circlearrowleft \mathrm{GRT}(k)$$

$$\widehat{\mathcal{P}a\mathcal{B}}(k) = \text{Malcev completion of } \mathcal{P}a\mathcal{B}$$

$$\mathcal{P}a\mathcal{B} \simeq \Pi_1(\mathbb{E}_2)$$

Kontsevich conjetured: GRT acts on Hochschild cohomology of a scheme

Theorem [V. Dolgushev, C. Rogers, T. Willwacher, 2015]

GRT acts as automorphisms on the Gerstenhaber algebra of polyvector fields

$$H^*(X, \mathcal{T}^\bullet_{\text{poly}}(X))$$

of a smooth complex variety X via contraction with the odd components of the Chern character of X.

Theorem [D. Calaque, M. Van den Bergh, 2010]

$$H^*(X, \mathcal{T}^\bullet_{\text{poly}}(X)) \xrightarrow{\text{HKR}\circ\text{Td}(X)^{1/2}} HH^*(X)$$

What am I doing?

DRW's proof relies on:

- Explicit Čech resolutions
- Identification of the Lie algebra of GRT with graph cohomology

Question

Can we find a conceptual proof of this theorem using techniques from operadic homotopy theory and DAG? Will this lead to new geometric examples of GRT actions?

Deligne's Conjecture on Hochschild cochains

A primer on operads

Definition

Let $\mathcal{C}$ be a closed symmetric monoidal category. An operad $\mathcal{O}$ in $\mathcal{C}$ is given by a collection of "n-ary operations" $\mathcal{O}(n) \in \mathcal{C}$ equipped with an action of the symmetric group S_n and equivariant insertion operations

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$

subject to unitality and associativity constraints.

Example

- $\mathcal{A}ssoc(n) = S_n$.
- $\mathcal{C}omm(n) = *$.

Algebras and modules

Definition

An algebra $A \in \mathcal{C}$ over an operad $\mathcal{O}$ is a morphism of operads

$$\mathcal{O} \to \mathcal{E}nd(A).$$

For any operad $\mathcal{O}$ there is a two-colored operad $M\mathcal{O}$ such that algebras over $M\mathcal{O}$ correspond to pairs (A, M) with $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and M an A-module over $\mathcal{O}$.

Homotopy theory of operads

Definition

Let $\mathcal{O}$ be an operad in a (well-behaved) monoidal model category. A homotopy $\mathcal{O}$-algebra is an algebra over a cofibrant replacement $\mathcal{O}_\infty \xrightarrow{\simeq} \mathcal{O}$.

Example

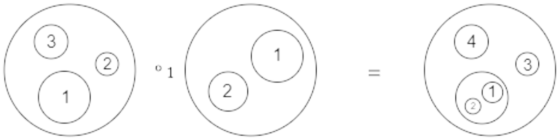
- Homotopy associative $= \mathbb{A}_\infty$
- Homotopy Lie $= L_\infty$

$\rightsquigarrow$ There is an ∞-categorical version of operads: ∞-operads

Little k-disks operads

$$\mathcal{A}ssoc \simeq \mathbb{E}_1 \subseteq \mathbb{E}_2 \subseteq \cdots \subseteq \mathbb{E}_\infty \simeq \mathcal{C}omm$$

Little 2-disks operad $\mathbb{E}_2$:



Homology of the little 2-disks operad

Symmetric monoidal model category/∞-category $\mathrm{Ch}(k)$ of complexes of k-modules with $k \supseteq \mathbb{Q}$.

$\mathcal{O}$ topological operad $\rightsquigarrow C_*(\mathcal{O})$ dg-operad with $C_*(\mathcal{O})(n) = C_*(\mathcal{O}(n); k)$

Theorem [F. Cohen, 1976]

$$H_*(\mathbb{E}_2) \simeq \mathcal{G}er$$

$\rightsquigarrow A$ a $C_*(\mathbb{E}_2)$-algebra $\Rightarrow H_*(A)$ a $\mathcal{G}er$-algebra.

Algebras over $\mathcal{G}er$

Degree 0 product

$$\cup : C_* \otimes_k C_* \to C_* \quad \text{associative and commutative}$$

Degree 1 "Lie bracket"

$$[-,-] : C_* \otimes_k C_* \to C_*[1] \quad \text{Jacobi and Leibnitz identity}$$

Deligne's Conjecture on Hochschild cochains

Proposition [M. Gestenhaber '62]

If A is an assoicative algebra, then the Hochschild cohomology $HH^*(A, A)$ has the structure of an algebra over $\mathcal{G}er$ with the cup product

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{p+q}) = (-1)^{pq} f(a_1 \otimes \cdots \otimes a_p) g(a_{p+1} \otimes \cdots \otimes a_{p+q})$$

and the Gerstenhaber bracket

$$[f, g]_G = f \circ g - (-1)^{(p-1)(q-1)} g \circ f.$$

Deligne's Conjecture 1993

There exists a natural action of the operad $C_*(\mathbb{E}_2)$ on the Hochschild complex $C^*(A, A)$ that descends to the canonical $\mathcal{G}er$-algebra structure on cohomology.

Many proofs

- Tamarkin ('98)
- Voronov ('99)
- McClure-Smith ('99)

} Construct operads $\simeq \mathbb{E}_2$ acting on $C^*(A, A)$

Conceptual explanation of why we should expect this action [Hu-Kriz-Voronov, 2005]:

$$C^*(A, A) \simeq \mathbb{R}\mathcal{H}om_{A \otimes A^{\mathsf{op}}}(A, A)$$

Eckmann-Hilton argument

Theorem

$\mathcal{C}$ be a closed monoidal, tensor unit I. Then

$$\mathcal{H}om(I, I) \in \mathcal{C}$$

is a commutative algebra.

$$I \xrightarrow{f} I \xrightarrow{g} I$$

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{f \otimes g} I \otimes I \xrightarrow{\cong} I.$$

∞-operadic Eckmann-Hilton argument

Dunn additivity theorem [J. Lurie, 2017]

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The morphism $\rho : \mathbb{E}_k \times \mathbb{E}_{k'} \to \mathbb{E}_{k+k'}$ given by taking product spaces of disks induces an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathcal{C}) \xrightarrow{\rho^*} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathcal{C})).$$

Corollary

The category of 2-algebras $\mathrm{Alg}_{\mathcal{A}ssoc}(\mathrm{Alg}_{\mathcal{A}ssoc}(\mathcal{C}))$ is equivalent to the category of $\mathbb{E}_2$-algebras $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C})$.

The derived center

$\rightsquigarrow$ Show that the Hochschild complex is an endomorphism object of a monoidal unit in a suitable monoidal ∞-category.

Slogan: The Hochschild cohomology is the derived center of the associative algebra.

Definition

Let $\mathcal{C}$ be a monoidal ∞-category. A center of $M \in \mathcal{C}$ is a final object $\mathfrak{z}(M)$ in $\mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{M\}$.

In particular, $\mathfrak{z}(M) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ is an algebra object in the underlying category.

The Hochschild complex as center

Proposition [J. Lurie, 2017]

Let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Then if the center of A exists, it is given by

$$\mathfrak{z}(A) \simeq \mathrm{Mor}_{\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})}(A, A) \in \mathcal{C}.$$

Theorem [F.]

Let A be an associative algebra viewed as object of $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Ch}(k))$. Then the Hochschild complex $C^*(A, A)$ is a center of A. In particular, $C^*(A, A) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Ch}(k))) \simeq \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Ch}(k))$ is an $\mathbb{E}_2$-algebra in chain complexes.

The Gerstenhaber operations of an $\mathbb{E}_2$-algebra

$\rightsquigarrow$ Have to show that this yields the correct Gerstenhaber algebra structure on homology.

$A \in \mathrm{Alg}_{C_*(\mathbb{E}_2)}(\mathrm{Ch}(k))$:

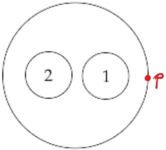
$$\underbrace{C_*(\mathbb{E}_2(2);k)}_{\cong C_*(S^1;k)} \otimes_k A^{\otimes_k 2} \to A$$

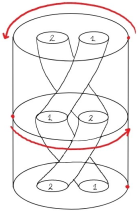
$$\Big\downarrow H_*(-)$$

$$\underbrace{H_*(S^1;k)}_{\cong k[0]\oplus k[1]} \otimes_k H_*(A)^{\otimes_k 2} \to H_*(A)$$

Given a 2-algebra, how does the product and bracket act in the $\mathbb{E}_2$-algebra structure induced by Dunn additivity?

Product $\sim$ choice of basepoint in $\mathbb{E}_2(2) \simeq S^1$
Bracket $\sim$ choice of loop in $\mathbb{E}_2(2) \simeq S^1$





Theorem [F.]

Let $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Ch}(k)))$ be a 2-algebra with products m_1 and m_2 and interchange law

$$\begin{array}{ccc} A \otimes A \otimes A \otimes A & \xrightarrow{m_2 \otimes m_2} & A \otimes A \\ {\scriptstyle m_1 \otimes m_1 \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})} \downarrow & \overset{h}{\Leftarrow} & \downarrow {\scriptstyle m_1} \\ A \otimes A & \xrightarrow[m_2]{} & A \end{array}$$

given by the 2-simplex $h : A^{\otimes_k 4} \to A[1]$ and let h' be the corresponding 2-simplex for the opposite multiplications. Then the bracket of the induced $\mathbb{E}_2$-algebra is given by the 2-simplex

$$h\iota_{2,3} + h'\iota_{1,4} + h'\iota_{2,3} + h\iota_{1,4},$$

where $\iota_{i,j} : A^{\otimes_k 2} \to A^{\otimes_k 4}$ is the appropriate inclusion.

Recovering the Gerstenhaber bracket

Corollary

The $\mathbb{E}_2$-algebra induced by the 2-algebra structure on $C^*(A, A)$ above carries the classical Gerstenhaber algebra structure in cohomology.

The Hochschild cohomology of a scheme

Hochschild cohomology of a scheme

$\rightsquigarrow$ Globalize this from k-algebras to (smooth separated finite type) schemes

Question: What is the correct Hochschild cohomology of X?

R. Swan:

$$HH^*(X) := \mathrm{Ext}^*_{\mathcal{O}_{X \times X}}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

$\rightsquigarrow$ Does not carry a natural Gerstenhaber bracket

Polydilfferential operators

M. Kontsevich: $HH^*(X) := \mathbb{H}^*(X, \mathcal{D}^*_{\text{poly}}(X))$, with

$$\mathcal{D}^*_{\text{poly}}(X)(\text{Spec}(A)) = D^*_{\text{poly}}(A) \subseteq C^*(A, A)$$

maps $A^{\otimes n} \to A$ that are differential operators in each argument.

$$\rightsquigarrow \mathcal{D}^*_{\text{poly}}(X) \in \text{Alg}_{\mathcal{G}er}(\mathcal{D}_\infty(\text{Sh}_k(X)))$$

I suggest that the most natural definition is to define the Hochschild cochain complex of X to be

$$\mathfrak{z}(\mathcal{O}_X) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}_\infty(\mathrm{Sh}_k(X))).$$

Comparison to the classical homotopy Gerstenhaber structure

$\rightsquigarrow$ Want to show that this agrees with the definition via polydifferential operators

Theorem [F.], 90% proven

The sheaf of polydifferntial operators is a center of $\mathcal{O}_X \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}_\infty(\mathrm{Sh}_k(X)))$.

Question: Does the induced $\mathbb{E}_2$-structure on $\mathcal{D}^*_{\mathrm{poly}}(X)$ agree with the one obtained by "classical" proofs of Deligne's conjecture?

Work in progress: Use formal geometry "Gelfand-Fuchs trick"