$\mathbb{E}_2$-algebra structures on the derived center of an algebraic scheme

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Abstract. This paper provides an explicit interface between J. Lurie's work on higher centers, and the Hochschild cohomology of an algebraic $\Bbbk$-scheme within the framework of deformation quantization. We first recover a canonical solution to Deligne's conjecture on Hochschild cochains in the affine and global cases, even for singular schemes, by exhibiting the Hochschild complex as a ∞-operadic center. We then prove that this universal $\mathbb{E}_2$-algebra structure precisely agrees with the classical Gerstenhaber bracket and cup product on cohomology in the affine and smooth cases. This last statement follows from our main technical result which allows us to extract the Gerstenhaber bracket of any $\mathbb{E}_2$-algebra obtained from a 2-algebra via Lurie's Dunn additivity theorem.

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1 Introduction

M. Kontsevich's formality theorem in deformation quantization states that the Hochschild-Kostant-Rosenberg map lifts to a morphism of homotopy Gerstenhaber algebras between polyvector fields and polydifferential operators of a smooth manifold. In [Tam03], D. Tamarkin found an algebraic proof of this theorem, extending it from manifolds to affine space $\mathbb{A}^n_{\Bbbk}$ where $\Bbbk$ is any field of characteristic zero by showing that the operad of little 2-disks is formal and using Deligne's conjecture on Hochschild cochains. This proof also highlights that these formality morphisms for the smooth Hochschild cochains are non-canonical, since Tamarkin's little disks formality depends on the choice of a Drinfeld associator.

In fact, using results from D. Bar-Natan [BN98], Tamarkin later showed in [Tam02] that his proof essentially identifies Drinfeld's associators with operad isomorphisms between the operad of parenthesized braids and the operad of parenthesized chord diagrams which are the identity on objects.

Based on Tamarkin's results, Kontsevich conjectured in [Kon03] that the Grothendieck-Teichmüller group (GT) should act on formality isomorphisms between $T_{\mathrm{poly}}(X)$ and $D_{\mathrm{poly}}(X)$ of a smooth complex variety, and that this action should be of motivic nature, arising as a consequence of the fact that the equations in the Knizhnik-Zamoldchikov associator are periods. This conjecture was proved by V. Dolgushev, C. Rogers and T. Willwacher in [DRW15]. They were able to show that the Deligne-Drinfeld elements of the Grothendieck-Teichmüller group act by contraction with the odd components of the Chern character of the variety on the cohomology of the sheaf of polyvector fields. In particular, they were able to give examples for which this action is non-trivial.

Dolgushev-Rogers-Willwacher use a result by D. Calaque and M. Van den Bergh in [CVdB10] which shows that the Kontsevich formality theorem can be extended to non-affine cases by adding a correction term to the HKR map which depends on the Atiyah class of the variety. Astonishingly, this correction term has the form $J^{1/2}$ with $J = \det(q(\mathrm{At}(X)))$ the Todd class of the variety and

$$q(x) = \frac{e^{x/2} - e^{-x/2}}{x}.$$

This is clearly reminiscent of (Kontsevich's generalization of) the classical Duflo isomorphism theorem, which states that for any finite dimensional Lie algebra $\mathfrak{g}$, we get an algebra isomorphism

$$\mathrm{PBW} \circ \det(q(\mathrm{ad}))^{1/2} : S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} U(\mathfrak{g})^{\mathfrak{g}}.$$

It is a result by A. Alekseev and C. Torossian [AT12] that the Grothendieck-Teichmüller group also acts on "classical" Duflo isomorphisms as above by changing the correction term. Unfortunately, this action was shown in [ABA00] to be trivial, but it nevertheless shows that there must be some deep connection between the Lie algebra case and the geometric case of the action on Duflo isomorphisms.

To this end, note that the codomain $U(\mathfrak{g})^{\mathfrak{g}} = H^*(\mathfrak{g}, U(\mathfrak{g}))$ is just the center $Z(\mathfrak{g})$ of the Lie algebra, and the cohomology of the sheaf of polydifferntial operators $\mathbb{H}^*(X, \mathcal{D}_{\mathrm{poly}}(X))$ of a smooth variety X computes the Hochschild cohomology of X, which is also commonly referred to as the "derived center". In a derived setting, we expect the center of an associative algebra object to be a 2-algebra, meaning it is equipped with two associative multiplications which are compatible up to homotopy, instead of a commutative algebra like in the classical case.

In [Lur17, Chapter 5], J. Lurie makes this idea precise by defining the center of an algebra over an ∞-operad. In particular, in case of the little k-disks ∞-operads $\mathbb{E}_k^{\otimes}$, he uses an ∞-categorical version of the Dunn additivity theorem to show that the higher center of an $\mathbb{E}_k$-algebra is indeed an $\mathbb{E}_{k+1}$-algebra. The idea is now to explain the Grothendieck-Teichmüller group actions above using B. Fresse's result on the connection between GT and the group of homotopy automorphisms of the rationalization of the topological little 2-disks operad, together with Lurie's result that the center of an associative algebra carries the structure of an $\mathbb{E}_2$-algebra.

In this paper we will lay the groundwork for proving this claim by connecting Lurie's work on higher centers with the classical results on the Hochschild cohomology of schemes. In particular, we will argue that the Hochschild complex of any quasi-compact separated scheme should be defined as its $\mathbb{E}_1$-operadic center, thereby equipping it with a canonical $\mathbb{E}_2$-algebra structure.

In what follows, given an ∞-category $\mathcal{C}$, we let $\mathrm{Alg}(\mathcal{C})$ denote the ∞-category of homotopy associative algebras in $\mathcal{C}$, and more generally $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ denotes the ∞-category of algebras over some ∞-operad $\mathcal{O}$. We let $\mathcal{D}(k)$ be the derived ∞-category of $\Bbbk$-modules. We denote the higher center of an algebra object A by $\mathfrak{z}(A)$.

We will first consider the affine case, and we will prove that we recover a solution to the classical Deligne conjecture. This is done in section 4.

Theorem A (Theorem 4.27)**.** *Let A be an associative $\Bbbk$-algebra. The Hochschild complex*

$$C^*(A, A) = \mathrm{Hom}_{\Bbbk}(A^{\otimes *}, A)$$

is a center for $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}_\infty(k))$. In particular, it is an element of $\mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}_\infty(k))) \cong \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}_\infty(k))$*. Its underlying Gerstenhaber algebra agrees with the classical Gerstenhaber algebra structure obtained from the $\mathcal{B}$races-algebra structure.*

In section 5 we recall a construction of the ∞-category $\mathrm{Sh}_\infty(X)$ of dg sheaves over a quasi-compact separated scheme X, and examine some basic properties of the center of the structure sheaf as $\mathbb{E}_1$-algebra in this ∞-category. We then go on to show that this version of Hochschild cochains has the desired local properties, even in the singular case. In particular, we show the following.

Theorem B (Theorem 5.46)**.** *Let $U = \operatorname{Spec}(A) \subseteq X$ be an affine open. The map $\mathbb{R}\Gamma_U : \operatorname{Sh}_\infty(X) \to \mathcal{D}_\infty(k)$ is lax symmetric monoidal and hence induces a map $\mathbb{R}\Gamma_U : \operatorname{Alg}_{\mathbb{E}_2}(\operatorname{Sh}_\infty(X)) \to \operatorname{Alg}_{\mathbb{E}_2}(\mathcal{D}_\infty(k))$. We have*

$$\mathbb{R}\Gamma_U(\mathfrak{z}_{\mathbb{E}_1}(\mathcal{O}_X)) \simeq \mathfrak{z}_{\mathbb{E}_1}(A).$$

In the smooth case, Kontsevich defined the Hochschild cochains to be the sheaf of polydifferential operators, as described above. This sheaf comes with the structure of a $\mathcal{B}races$-algebra, just like in the local case, and therefore a homotopy Gerstenhaber algebra by Tamarkin's results. In order to compare to the existing GT actions, we need to compare our newly obtained $\mathbb{E}_2$-algebra structure to this homotopy Gerstenhaber algebra structure.

Theorem C (Theorem 5.54, Theorem ...)**.** *For a smooth quasi-compact separated scheme X of finite type, the sheaf of polydifferential operators $\mathcal{D}_{\mathrm{poly}}(X)$ is a center of $\mathcal{O}_X$*

$$\mathcal{D}_{\mathrm{poly}}(X) \simeq \mathfrak{z}_{\mathbb{E}_1}(\mathcal{O}_X).$$

This equips $\mathcal{D}_{\mathrm{poly}}(X)$ with the structure of an $\mathbb{E}_2$-algebra, and after fixing a Drinfeld associator a $\mathcal{G}er_\infty$-algebra. The corresponding Gerstenhaber algebra in the $\Bbbk$-linear derived 1-category $\operatorname{HoSh}_\infty(X)$ agrees with the classical one coming from the $\mathcal{B}races$-algebra structure.

In the course of proving these results, we also obtain a couple of technical results about higher centers. In particular, we show how to explicitly obtain the Gerstenhaber bracket of a 2-algebra up to homotopy using Lurie's version of the Dunn additivity theorem (see 3.18). We also show that the 2-algebra structure on a center which is obtained as an endomorphism object indeed corresponds to the composition product (Yoneda product) and the convolution product (see **??**). In a similar manner, we further obtain a stability result stating that the space of Gerstenhaber algebra structures on a center is contractible, which we expect will be helpful in examining action of the Grothendieck-Teichmueller group later on.

Related work. The connection between the Hochschild cochain complex of an associative algebra A and its higher center, i.e. the universal $\mathbb{E}_2$-algebra acting on it, has been known already to Kontsevich in [Kon03]. Similarly, when Deligne made his conjecture he already stated that he expected the little 2-disk algebra structure to come from the composition and convolution product via some type of Eckmann-Hilton argument. In fact, Hu-Kriz-Voronov in [HKV06] proved a simplicial version of Deligne's conjecture using this idea and explicit models of the little disks operads. The issue with both these ideas has been the difficulty to state and use them rigorously using only the language of model categories and derived 1-categories available at the time.

After Lurie published his theory of higher centers and in particular his version of the Dunn additivity theorem in his DAG papers, it became clear how to state the universal property identified by Kontsevich and what the correct higher replacement of the Eckmann-Hilton argument is. In this sense, the theory of quasi-categories and ∞-operads is essential to this paper and its related work.

In 2013, John Francis used Lurie's Dunn additivity theorem to examine centers of stable ∞-categories and to relate the center to the module of derivations, which was also explained already by

Kontsevich in [Kon03]. He also defines the Hochschild cohomology of an algebra over an ∞-operad $\mathcal{O}$ as the hom set of $\mathcal{O}$-module maps over A from A to itself, which is closely related to Lurie's definition of the center.

In 2020, Iwanari [Iwa20] used this definition of Hochschild cohomology to show that pair of Hochschild cohomology and homology of a linear category over some commutative ring spectrum gives an algebra over the KS operad which is a generalization of the $\mathbb{E}_2$-operad.

In 2023, Brav and Rozenblyum [BR23] proved a cyclic version of Deligne's conjecture (i.e. replacing the little 2-disk operad by the framed version) also using the above described techniques. In particular, their method also relies on Lurie's version of Dunn additivity. However, non of the above papers proves a comparison to the classical solutions of Deligne's conjecture, or make any claim about the underlying Gerstenhaber algebra structure. Similarly, the author is not aware of any comparison of the center of a scheme to the classical sheaf of polydifferential operators.

Conventions. Throughout this paper $\Bbbk$ is a field of characteristic zero. The term "operad" is reserved for non-reduced unital symmetric operads. Complexes are generally chain graded unless states otherwise, and we view non-negatively graded cochain complexes as non-positive chain complexes. We denote the presheaf tensor product simply by "$\otimes$", and we decorate symbols with "$(-)^a$" to indicate sheafification. We try to use the term "∞-category" for ∞-categories, but if nothing else is stated "category" refers to ∞-category and "1-category" refers to ordinary categories.

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2 Preliminaries

We follow J. Lurie's formalism for ∞-operads as developed in [Lur17]. In particular, an ∞-operad is a morphism $p : \mathcal{O}^{\otimes} \to Fin_*$ of ∞-categories satisfying a list of conditions making $\mathcal{O}^{\otimes}$ into an ∞-category of operators. We will freely use notation from [Lur17] regarding ∞-operads and algebras and modules over these. We will also give a quick overview of the history of Deligne's conjecture on Hochschild cochains.

2.1 Morphism objects and operadic centers

We review Lurie's theory of morphism objects and operadic centers. The definitions we use can be found in [Lur17, Section 4.2, 4.7 and 5.3].

Definition 2.1. Let $\mathfrak{a}$ and $\mathfrak{m}$ be the two colors of the operad **LM**. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. We then say that q exhibits the ∞-category $\mathcal{M} := \mathcal{C}_{\mathfrak{m}}$ as left tensored over the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}^{\otimes} := \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathrm{Assoc}^{\otimes}$. In particular, q determines a tensoring

$$\otimes : \mathcal{C}_{\mathfrak{a}} \times \mathcal{M} \to \mathcal{M}$$

well-defined up to homotopy that is compatible with the monoidal structure on $\mathcal{C}_{\mathfrak{a}}$ up to homotopy.

Definition 2.2. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ exhibit $\mathcal{M}$ as left tensored over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Then we denote by

$$\mathrm{LMod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{LM}^{\otimes}}(\mathcal{C})$$

the ∞-category of pairs of associative algebras in $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ and left modules.

In particular, any monoidal ∞-category is canonically left tensored over itself.

Recall that for ordinary categories, internal homs and more generally enrichments are right adjoint to a tensoring. Similarly, one makes the following definition.

Definition 2.3. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads exhibiting $\mathcal{M}$ as left tensored over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. If $M, N \in \mathcal{M}$, a morphism object for M and N is an object $\mathrm{Mor}(M, N) \in \mathcal{C}_{\mathfrak{a}}$ together with a map $\rho \in \mathrm{Map}_{\mathcal{M}}(C \otimes M, N)$ such that for each $C \in \mathcal{C}_{\mathfrak{a}}$, post-composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{a}}}(C, \mathrm{Mor}(M, N)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{M}}(C \otimes M, N). \tag{1}$$

We call $\mathcal{M}$ enriched over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ if all the morphisms objects exist.

The following result shows that we can think of morphism objects as the classifying object of action maps $A \otimes M \to N$ with $A \in \mathcal{C}_{\mathfrak{a}}$.

Proposition 2.4. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Let $M, N \in \mathcal{M}$. Then an object $\mathrm{Mor}(M, N) \in \mathcal{C}_{\mathfrak{a}}$ together with a map $\alpha : \mathrm{Mor}(M, N) \otimes M \to N$ is a morphism object of M and N if and only if $(\mathrm{Mor}(M, N), \alpha) \in \mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/N}$ with map given by $- \otimes M : \mathcal{C}_{\mathfrak{a}} \to \mathcal{M}$ is final.*

Proof. Note that $\mathcal{M}_{/N} \to \mathcal{M}$ is a right fibration, and since these are stable under base change, so is $f : \mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/N} \to \mathcal{C}_{\mathfrak{a}}$. Consider the functor $F : \mathcal{C}_{\mathfrak{a}}^{\mathrm{op}} \to \mathbf{An}$ classifying f. Then by [Lur12, lemma 2.2.2.4], its underlying functor $hF : h\mathcal{C}_{\mathfrak{a}}^{\mathrm{op}} \to \mathcal{H}$ can be recovered as follows. On objects, an object $X \in \mathcal{C}_{\mathfrak{a}}$ is sent to its fiber

$$(\mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/N}) \times_{\mathcal{C}_{\mathfrak{a}}} \{X\} \simeq \{X \otimes M\} \times_{\mathcal{M}} \mathcal{M}_{/N} \simeq \mathrm{Map}_{\mathcal{M}}(X \otimes M, N).$$

Given a morphism $e : Y \to X \in \mathrm{Mor}_{\mathcal{C}_{\mathfrak{a}}}(Y, X)$ in $h\mathcal{C}_{\mathfrak{a}}$, the induced map between the fibers comes from solving the lifting problem

$$\begin{tikzcd}
\{1\} \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N) \arrow[r, hook] \arrow[d, hook] & \mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/N} \arrow[d, "f"] \\
\Delta^1 \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N) \arrow[ur, dashed] \arrow[dr] & \mathcal{C}_{\mathfrak{a}} \\
& \Delta^1 \arrow[u, "e"']
\end{tikzcd},$$

and restricting the lift to $\{0\} \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N)$. Since f is a pullback of the right fibration $\mathcal{M}_{/N} \to \mathcal{M}$, the lift above is induced by the solution to

$$\begin{tikzcd}
\{1\} \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N) \arrow[r] \arrow[d] & \mathcal{M}_{/N} \arrow[d] \\
\Delta^1 \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N) \arrow[r, "e\otimes \mathrm{id}_M"'] \arrow[ur, dashed] & \mathcal{M}
\end{tikzcd}.$$

But the restriction of this lift to $\{0\} \times \mathrm{Map}_{\mathcal{M}}(X \otimes M, N)$ is given by pre composition with $e \otimes \mathrm{id}_M$. Therefore, we see that hF is given by the composition of $- \otimes M$ and $\mathrm{Map}_{\mathcal{M}}(-, N)$. Recall that by [Lur12, prop. 4.4.4.5], an object $(X, X \otimes M \xrightarrow{\eta} N)$ is final in $\mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/N}$ if and only if the pair $(X, \eta \in hF(X))$ represents hF. Then we are done after noting that by definition, $(\mathrm{Mor}(M, N), \alpha)$ is a morphism object exactly if it represents the functor $X \mapsto \mathrm{Map}_{\mathcal{M}}(X \otimes M, N)$. $\square$

Now put $N = M$. Then by the above proposition, $\mathrm{End}(M) := \mathrm{Mor}(M, M) \in \mathcal{C}_{\mathfrak{a}}$ classifies actions of elements of $\mathcal{C}_{\mathfrak{a}}$ on M. Note however that these are plain actions $A \otimes M \to M$ that do not require A to be an algebra object and do not require M to satisfy the axioms of a module over A. However, we do expect $\mathrm{End}(M)$ to carry the structure of an associative algebra coming from composition, and M to be a module over it.

Definition 2.5. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Define the endomorphism ∞-catetgory of $M \in \mathcal{M}$ as

$$\mathcal{C}_{\mathfrak{a}}[M] = \mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/M}.$$

We show in appendix A that $\mathcal{C}_{\mathfrak{a}}[M]$ agrees with Lurie's endomorphism ∞-category as in [Lur17, Definition 4.7.1.1]. In particular, it admits the structure of a monoidal ∞-category with the tensor product given up to homotopy by

$$(A, A \otimes M \to M) \otimes (B, B \otimes M \to M) = (A \otimes B, A \otimes B \otimes M \to A \otimes M \to M).$$

Then by [Lur17, Corollar 3.2.2.5] with $K = \emptyset$ and $\mathcal{O}^{\otimes} = \mathrm{Assoc}^{\otimes}$ we automatically get the following.

Corollary 2.6. *Assume the ∞-category $\mathcal{C}_{\mathfrak{a}}[M]$ has a final object $(End(M), \alpha)$. Then $(End(M), \alpha)$ can be promoted to an object of $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}[M])$ in an essentially unique way. We denote this object again by $\mathrm{End}(M)$. Note that $\mathrm{End}(M)$ is automatically final in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}[M])$.*

Up to homotopy, the tensor product of $\mathrm{End}(M)$ with itself is given by the action

$$(\mathrm{End}(M) \otimes \mathrm{End}(M)) \otimes M \simeq \mathrm{End}(M) \otimes (\mathrm{End}(M) \otimes M) \xrightarrow{\mathrm{id}_{\mathrm{End}(M)} \otimes \alpha} \mathrm{End}(M) \otimes M \xrightarrow{\alpha} M,$$

and hence the algebra structure on $\mathrm{End}(M)$ has a multiplication making the following diagram commute

$$\begin{tikzcd}
 & \mathrm{End}(M) \otimes M \arrow[dr] & \\
(\mathrm{End}(M) \otimes \mathrm{End}(M)) \otimes M \arrow[ur] \arrow[rr] & & M
\end{tikzcd}.$$

Further, M is automatically a module over the algebra $\mathrm{End}(M)$, and this action $\mathrm{End}(M) \otimes M \to M$ is universal among algebra actions on M as a module.

Corollary 2.7. *In the above situation, we have an equivalence of ∞-categories*

$$\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}[M]) \to \mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

Proof. This follows directly from [Lur17, Theorem 4.7.1.34] and the results of chapter 4.1.3 in [Lur17]. □

There are a variety of interesting situations in which such an (end)omorphism object fails to exist, in particular if we consider ∞-categories arising as categories of algebra objects. The archetypal example is the following.

Example 2.8. Let k be a field and consider the (symmetric) monoidal category Alg_k as left tensored over itself. Let $A \in \mathcal{C}$ be some k-algebra. Then for any endomorphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, M)$, the pair $(k, k \otimes M \cong M \xrightarrow{\varphi} M)$ is an object of $\mathcal{C}[M]$. Therefore, if $(A, \eta) \in \mathcal{C}[M]$ were a final object, then $\eta \circ (u_A \otimes \mathrm{id}_M) = \varphi$ for any endomorphism $\varphi : M \to M$. But this not possible unless $M = k$ is the trivial k-algebra. This was to be expected, since we know that the monoidal category of k-algebras is not closed.

The solution to this problem is to relax our expectations on the morphism object. In the above discussion, we start out requiring that $\mathrm{Mor}(M, N)$ classify all (plain) actions $C \otimes M \to N$, and then in the case $N = M$ get for free that $\mathrm{End}(M)$ also classifies algebra actions of algebras on M. Instead, we now consider objects that only classify the algebra actions.

Definition 2.9. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibraiton of ∞-operads, and let $M \in \mathcal{M}$. A center $\mathfrak{Z}(M)$ of M is a final object of $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$. We generally identify $\mathfrak{Z}(M)$ with its image in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$.

Clearly if M admits an endomorphism object, then this endomorphism object is also a center of M. The converse does not hold: The category $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}[M])$ might have final objects although $\mathcal{C}_{\mathfrak{a}}[M]$ does not.

Example 2.10 (Example 2.8 continued)**.** The ordinary center $Z(A)$ of an associative k-algebra A is indeed the universal algebra object acting on A. To see this, note first that the center is a commutative algebra, and therefore an algebra object in the category of associative algebras. It comes with a natural action on A given by multiplication in A. Now suppose that B is a commutative algebra with action $\eta : B \otimes A \to A$ making A into a B-module. Then the restriction of η to A yields the identity on A and η must be an algebra morphism. Hence

$$\begin{aligned} \eta(b \otimes a) &= \eta(b \otimes 1) \cdot \eta(1 \otimes a) = \eta(b \otimes 1) \cdot a \quad \text{and} \\ \eta(b \otimes a) &= \eta(1 \otimes a) \cdot \eta(b \otimes 1) = a \cdot \eta(b \otimes 1), \end{aligned}$$

showing that η sends B to $Z(A)$.

There also is a relative version of the center.

Definition 2.11. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads, let $\mathbb{1}$ denote the monoidal unit of $\mathcal{C}_{\mathfrak{a}}^{\otimes}$, and let $f : M \to N$ be a morphism in $\mathcal{M}$. A centralizer $\mathfrak{z}(f)$ of f is a final object in

$$\mathrm{Act}(f) := (\mathcal{C}_{\mathfrak{a}})_{\mathbb{1}/} \times_{\mathcal{M}_{M/}} \mathcal{M}_{M//N}.$$

We generally identify $\mathfrak{z}(f)$ with its image in $\mathcal{C}_{\mathfrak{a}}$.

The objects of this ∞-category are given by commuting triangles in $\mathcal{M}$

$$\begin{tikzcd} & C \otimes M \arrow[dr] & \\ \mathbb{1} \otimes M \arrow[ur] \arrow[rr] & & N \end{tikzcd}.$$

In particular, the centralizer is equipped with an action $\mathfrak{z}(f) \otimes M \to N$ making the above diagram commute.

Lemma 2.12. *Let $f : M \to N$ be a morphism in $\mathcal{M}$. Let $\overline{M} \in \mathrm{LMod}_{\mathbb{1}}(\mathcal{M})$ be a lift of M as module over the trivial algebra. Let $\mathcal{C}^{\otimes}_{\overline{M}_{\mathcal{LM}/}} \to \mathcal{LM}^{\otimes}$ be defined as in [Lur17, Definition 4.2.1.28]. Then centralizers of f can be identified with morphism objects*

$$\mathrm{Mor}_{\mathcal{M}_{M/}}(\mathrm{id}_M, f) \in (\mathcal{C}_{\mathfrak{a}})_{\mathbb{1}/}.$$

Proof. Let $\mathcal{C}'^{\otimes} := \mathcal{C}^{\otimes}_{\overline{M}_{\mathcal{LM}}}$. By proposition 2.4, it suffices to show that we have an equivalence of ∞-categories $\mathrm{Act}(f) \simeq (\mathcal{C}')_{\mathfrak{a}} \times_{\mathcal{M}'} \mathcal{M}'_{/f}$. But we have $(\mathcal{C}')_{\mathfrak{a}} \times_{\mathcal{M}'} \mathcal{M}'_{/f} \simeq (\mathcal{C}_{\mathfrak{a}})_{\mathbb{1}/} \times_{\mathcal{M}_{M/}} (\mathcal{M}_{M/})_{/N}$, so this is clear. $\square$

We would like to see that these notions are compatible, in the sense that the centralizer of an identity morphism recovers the center.

Proposition 2.13 (Proposition 5.3.1.8 [Lur17])**.** *Let $M \in \mathcal{M}$, and suppose there exists a centralizer $\mathfrak{z}(\mathrm{id}_M) \in \mathcal{C}_{\mathfrak{a}}$. Then there exists a center $\mathfrak{z}(M) \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$. Further, a lift of M to a module over an algebra $A \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$ exhibits A as a center of M if and only if the action map $A \otimes M \to M$ exhibits A as a centralizer of id_M.*

Proof. Recall that the centralizer of the identity is a morphism object $\mathrm{Mor}_{\mathcal{M}_{M/}}(\mathrm{id}_M, \mathrm{id}_M)$. By corollary 2.6, this morphism object admits an essentially unique structure of an algebra object in $(\mathcal{C}_{\mathfrak{a}})_{\mathbb{1}/}$, and id_M lifts to a module over this algebra structure. In particular, $\mathfrak{z}(\mathrm{id}_M)$ admits a canonical algebra structure making it into the center of id_M in $\mathcal{M}_{M/}$. Now use [Lur17, Lemma 5.3.1.10] to see that the forgetful functor $\mathrm{LMod}(\mathcal{M}_{M/}) \times_{\mathcal{M}_{M/}} \{\mathrm{id}_M\} \to \mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ preserves final objects. $\square$

2.2 Tensor product of ∞-operads

The Boardman-Vogt tensor product on ordinary operads is designed such that algebras over the tensor product $\mathcal{P} \boxtimes_{\mathrm{BV}} \mathcal{O}$ are given by $\mathcal{P}$-algebras in the category of $\mathcal{O}$-algebras. We briefly review the corresponding construction for ∞-operads.

We want to capture bilinearity of a map between ∞-operads. To this end, define a functor $\wedge : \mathit{Fin}_* \times \mathit{Fin}_* \to \mathit{Fin}_*$ by sending $(\langle m \rangle, \langle n \rangle)$ to the pointed set $(\langle m \rangle^{\circ} \times \langle n \rangle^{\circ})_+ \cong \langle mn \rangle$, where the isomorphism is given by the lexicographic ordering, and by sending $(f : \langle m \rangle \to \langle n \rangle, g : \langle m' \rangle \to \langle n' \rangle)$ to

$$\langle mm' \rangle \xrightarrow{\cong} (\langle m \rangle^{\circ} \times \langle m' \rangle^{\circ})_+ \xrightarrow{f \times g} (\langle n \rangle^{\circ} \times \langle n' \rangle^{\circ})_+ \xrightarrow{\cong} \langle nn' \rangle.$$

We call a map of simplicial sets $F : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ a bifunctor of ∞-operads if the diagram below commutes, and if F sends pairs of inert maps to an inert map in $\mathcal{O}''^{\otimes}$.

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \arrow[r, "F"] \arrow[d] & \mathcal{O}''^{\otimes} \arrow[d] \\ Fin_* \times Fin_* \arrow[r, "\wedge"] & Fin_* \end{tikzcd}$$

In particular, define $\mathrm{Bil}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ to be the full subcategory of $\mathrm{Fun}_{Fin_*}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$ spanned by the bifunctors.

We claim that for a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the ∞-category of bifunctors from $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}$ to $\mathcal{C}^{\otimes}$ is equivalent to the appropriate ∞-category of $\mathcal{O}$-algebras in the symmetric monoidal ∞-category $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$. To this end, recall the symmetric monoidal structure on the ∞-category $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$. This is supposed to capture the fact that the tensor product in $\mathcal{C}$ descends to a tensor product of $\mathcal{O}'$-algebras in $\mathcal{C}$. We define a map of simplicial sets $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to Fin_*$ by the following universal property. If $K \to Fin_*$ is a map of simplicial sets, then there is a natural bijection between $\mathrm{Hom}_{Fin_*}(K, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes})$ and the set of diagrams

$$\begin{tikzcd} K \times \mathcal{O}'^{\otimes} \arrow[r, "F"] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d] \\ Fin_* \times Fin_* \arrow[r, "\wedge"] & Fin_* \end{tikzcd}$$

such that for $v \in K$ a vertex and f an inert morphisms in $\mathcal{O}'^{\otimes}$, the map $F(s_0(v), f)$ is inert in $\mathcal{C}^{\otimes}$. In particular, the fiber $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}_{\langle 1 \rangle}$ over $\langle 1 \rangle \in Fin_*$ is given by the full subcategory of $\mathrm{Fun}_{Fin_*}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ of maps that preserve inert morphisms, and hence can be identified with the ∞-category $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$. Fixing $\langle n \rangle \in Fin_*$, we get a new map $\mathcal{O}'^{\otimes} \to Fin_*$ given by the following diagram

$$\begin{tikzcd} \mathcal{O}'^{\otimes} \arrow[rr] \arrow[d, "\cong"'] & & Fin_* \\ \Delta^0 \times \mathcal{O}'^{\otimes} \arrow[r] & Fin_* \times Fin_* \arrow[ur, "\wedge"'] \end{tikzcd}$$

were the lower horizontal map picks out $\langle n \rangle \in Fin_*$. We can informally describe this map as $\langle n \rangle \wedge p$, if $p : \mathcal{O}'^{\otimes} \to Fin_*$ is the map making $\mathcal{O}'^{\otimes}$ an ∞-operad. It particular, it sends $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ to $\langle nm \rangle = \langle n \rangle \wedge p(X)$ and $f : X \to Y$ to $\mathrm{id}_{\langle n \rangle} \wedge p(f)$. Hence the fiber $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}_{\langle n \rangle}$ is given by the full subcategory of $\mathrm{Fun}_{Fin_*}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ sending inert maps to inert maps, where now the map $\mathcal{O}'^{\otimes} \to Fin_*$ is given by this new map $\langle n \rangle \wedge p$. If $F : \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes} \in \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}_{\langle n \rangle}$, and $X \in \mathcal{O}'$, then $F(X) \in \mathcal{C}^{\otimes}_{\langle n \rangle}$ can be described by a tuple $(F(X)_1, \dots, F(X)_n) \in \mathcal{C}^n$. This argument shows that F can indeed be identified with a tuple $(F_1, \dots, F_n) \in \mathrm{Alg}_{\mathcal{O}'}(C)^n$. By [Lur17, 3.2.4.3] the map $q' : \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to Fin_*$ is a coCartesian fibration and a morphism f in $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ is q'-coCartesian if and only if for every $X \in \mathcal{O}'$, the image $f(X)$ is q-coCartesian in $\mathcal{C}^{\otimes}$. That means that $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ is again a symmetric monoidal ∞-category. For $X \in \mathcal{O}'$, the evaulation map $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is a morphism of ∞-operads, hence a lax symmetric monoidal functor, and we see that the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ is given by the pointwise tensor product on $\mathcal{C}^{\otimes}$.

Now an $\mathcal{O}$-algebra in $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ is given by a morphism of simplicial sets $\mathcal{O}^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ over Fin_* sending inert morphisms to inert morphisms. In particular, by the construction of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$, such an $\mathcal{O}$-algebra is given by a diagram

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d] \\
Fin_* \times Fin_* \arrow[r, "\wedge"] & Fin_*
\end{tikzcd}$$

such that for every $X \in \mathcal{O}^{\otimes}$ and every inert map f in $\mathcal{O}'^{\otimes}$, the tuple (id_X, f) is sent to an inert map in $\mathcal{C}^{\otimes}$. The condition that inert morphisms in $\mathcal{O}^{\otimes}$ are sent to inert maps in $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ translates to the fact that for inert maps f in $\mathcal{O}^{\otimes}$ and $X \in \mathcal{O}'$, the tuple (f, id_X) is sent to an inert map in $\mathcal{C}^{\otimes}$; and together those two conditions say exactly that tuples of inert maps are sent to an inert map. But this is clearly the same as a bifunctor $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$.

We now say that a bifunctor $F : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ if for every ∞-operad $\mathcal{C}^{\otimes}$, precomposition with F determines an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to \mathrm{Bil}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}).$$

In particular, in this case, if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, the above discussion shows that we have an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})).$$

In this sense, the tensor product of ∞-operads is a derived version of the Bordmann-Vogt tensor products of operads.

2.3 The little cubes operads

Consider the topological (one-colored) operad $\mathbb{E}_k^T$ with $\mathbb{E}_k^T(n) = \mathrm{Rect}(\square^k \times \{1, \dots, n\}, \square^k)$ the space of rectilinear embeddings. Let $\mathbb{E}_k^{T,\otimes}$ denote its category of operators, and consider the corresponding simplicial category $\mathrm{Sing}_\bullet(\mathbb{E}_k^{T,\otimes})$. Taking the homotopy coherent nerve, we obtain an ∞-category $\mathbb{E}_k^{\otimes}$, which is an ∞-operad since the underlying simplicial operad is fibrant. In particular, objects in $\mathbb{E}_k^{\otimes}$ are given by $\langle n \rangle$ for $n \in \mathbb{N}$, morphisms are given by points in

$$\mathrm{Map}_{\mathbb{E}_k^{\otimes}}(\langle n \rangle, \langle m \rangle) = \coprod_{f : \langle n \rangle \to \langle m \rangle} \prod_{j \in \langle m \rangle^{\circ}} \mathrm{Rect}(\square^k \times \{1, \dots, n\}, \square^k),$$

and to give a 2-simplex with boundary as shown below is equivalent to giving a path from $F \circ E$ to G in $\mathrm{Map}_{\mathbb{E}_k^{\otimes}}(\langle m \rangle, \langle k \rangle)$.

$$\begin{tikzcd}
& \langle m \rangle \arrow[dr, "F"] & \\
\langle n \rangle \arrow[ur, "E"] \arrow[rr, "G"'] & & \langle k \rangle
\end{tikzcd}$$

For $k = 1$ we get the $\mathbb{E}_1^{\otimes}$-operad, which is equivalent to the ∞-operad $\mathrm{Assoc}^{\otimes}$ and governs homotopy associative algebras. For $k = 2$, we instead recover the ∞-operadic version of the little 2-disks operad D_2. In particular, an element in $\mathrm{Mul}_{\mathbb{E}_1}(\langle n \rangle, \langle 1 \rangle)$ is given by a rectangular embedding of n copies of the interval $[0, 1]$ into the interval $[0, 1]$. An element in $\mathrm{Mul}_{\mathbb{E}_2}(\langle n \rangle, \langle 1 \rangle)$ is given by a rectangular embedding of n copies of the square $[0,1] \times [0,1]$ into the square $[0,1] \times [0,1]$.

Note that for $k \geq 0$, we have a homotopy equivalence $\mathrm{Mul}_{\mathbb{E}_k}(\langle 2 \rangle, \langle 1 \rangle) \simeq S^{k-1}$ given by drawing a line between the middle points of the two copies of $[0,1]^k$ inside $[0,1]^k$ with direction given by going from the label 2 to the label 1, and finding the intersection of this line with the boundary of $[0,1]^k$ in the positive direction. Then one can deform the boundary of $[0,1]^k$ into an S^{k-1} and get the corresponding point there. We frequently use this homotopy equivalence as a convenient method to label morphisms in the little cubes operads. In particular, fix a homotopy inverse $S^0 \to \mathrm{Mul}_{\mathbb{E}_1}(\langle 2 \rangle, \langle 1 \rangle)$ and a homotopy inverse $S^1 \to \mathrm{Mul}_{\mathbb{E}_2}(\langle 2 \rangle, \langle 1 \rangle)$. Then we get two elements in $\mathrm{Mul}_{\mathbb{E}_1}(\langle 2 \rangle, \langle 1 \rangle)$ named μ_0 and μ_π, and for every $t \in [0, 2\pi)$ we get an element $\mu_t \in \mathrm{Mul}_{\mathbb{E}_2}(\langle 2 \rangle, \langle 1 \rangle)$.

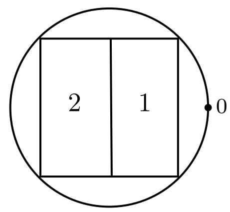
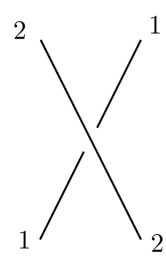
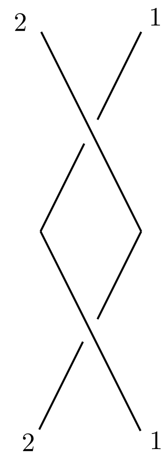


Figure 1: The element $\mu_0 \in \mathrm{Mul}_{\mathbb{E}_2}(\langle 2 \rangle, \langle 1 \rangle)$.

Recall that 2-morphisms in $\mathbb{E}_k^\otimes$ are given by paths in the relevant hom-spaces. There are two such 2-morphisms in $\mathbb{E}_1^\otimes$ that will play a special role in the subsequent discussion. On the one hand, for each $t \in [0, 2\pi)$, there is a 2-morphisms $\sigma_t \in \mathrm{Map}_{\mathbb{E}_2^\otimes}(\langle 2 \rangle, \langle 1 \rangle)_1$ with boundary given by t and $t + \pi (\mathrm{mod}\ 2\pi)$ that is represented by the braid



On the other hand, for each $t \in [0, 2\pi)$ there is a non-trivial 2-morphism $\gamma_t \in \mathrm{Map}_{\mathbb{E}_2^\otimes}(\langle 2 \rangle, \langle 1 \rangle)_1$ between t and itself represented by the composition of braids



The classical Eckmann-Hilton argument shows that in the 1-categorical case algebra objects in the category of algebra objects yield commutative algebra objects. In particular, if $(A, \cdot)$ is an algebra in a symmetric monoidal category $\mathcal{C}$ and $* : (A, \cdot) \otimes (A, \cdot) \to (A, \cdot)$ endows $(A, \cdot)$ with the structure of an algebra object in the category of algebras in $\mathcal{C}$, then both operations $\cdot$ and $*$ agree and they are commutative. In this sense, an $\mathbb{E}_1$-algebra inside the symmetric monoidal category of $\mathbb{E}_1$-algebras in $\mathcal{C}$ is the same as a commutative algebra inside $\mathcal{C}$, which in the 1-categorical case is the same as an $\mathbb{E}_2$-algebra. In fact, this pattern continues for all the little k-cubes operads, as was shows by Dunn for topological operads and later by Lurie for ∞-operads. To explain this, for $k, k' \geq 0$, define a topological functor

$$\rho : \mathbb{E}_k^{T,\otimes} \times \mathbb{E}_{k'}^{T,\otimes} \to \mathbb{E}_{k+k'}^{T,\otimes}$$

given on objects by $\rho(\langle m \rangle, \langle n \rangle) = \langle m \rangle \wedge \langle n \rangle$, and sending a pair of morphisms $(\alpha, \{f_j : \square^k \times \alpha^{-1}(\{j\}) \to \square^k\}_{j \in \langle n \rangle^\circ})$ and $(\beta, \{g_i : \square^{k'} \times \beta^{-1}(\{i\}) \to \square^{k'}\}_{i \in \langle n' \rangle^\circ})$ to

$$(\alpha \wedge \beta, \{f_j \times g_i : \square^{k+k'} \times \alpha^{-1}(\{j\}) \times \beta^{-1}(\{i\}) \to \square^{k+k'}\}_{j \in \langle n \rangle^\circ, i \in \langle n' \rangle^\circ}).$$

In order for this to make sense, we note that viewing a tuple $(j, i) \in \langle n \rangle^\circ \times \langle n' \rangle^\circ$ as an element of $\langle nn' \rangle^\circ$, we have $(\alpha \wedge \beta)^{-1}((j, i)) = \alpha^{-1}(\{j\} \times \beta^{-1}(\{i\})$. This descends to a simplicial functor, and then taking the homotopy coherent nerve to a map of ∞-categories $\rho : \mathbb{E}_k^\otimes \times \mathbb{E}_{k'}^\otimes \to \mathbb{E}_{k+k'}^\otimes$. By construction, the diagram

$$\begin{array}{ccc} \mathbb{E}_k^\otimes \times \mathbb{E}_{k'}^\otimes & \xrightarrow{\rho} & \mathbb{E}_{k+k'}^\otimes \\ \downarrow & & \downarrow \\ Fin_* \times Fin_* & \xrightarrow{\wedge} & Fin_* \end{array}$$

commutes, and clearly ρ sends pairs of inert morphisms to inert morphisms. Thus, ρ is a bifunctor of ∞-operads.

Theorem 2.14 (Dunn additivity, Theorem 5.1.2.2 [Lur17])**.** *The bifunctor $\rho : \mathbb{E}_k^\otimes \times \mathbb{E}_{k'}^\otimes \to \mathbb{E}_{k+k'}^\otimes$ exhibits the ∞-operad $\mathbb{E}_{k+k'}^\otimes$ as a tensor product of $\mathbb{E}_k^\otimes$ and $\mathbb{E}_{k'}^\otimes$.*

This implies that for every symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, precomposition with ρ determines an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathcal{C})).$$

In particular, this is essentially surjective, meaning that for every $\mathbb{E}_k$-algebra A in $\mathbb{E}_{k'}$-algebras in $\mathcal{C}^{\otimes}$, there exists a $\mathbb{E}_{k+k'}$-algebra $\tilde{A}$ in $\mathcal{C}^{\otimes}$ such that $\tilde{A} \circ \rho$ is equivalent to A in the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathcal{C}))$.

2.4 Deligne's conjecture on Hochschild cochains

Let A be an associative k-algebra. The category of left modules over the algebra $A \otimes A^{\mathrm{op}}$ is isomorphic to the category of A-bimodules. This is a monoidal category with tensor unit given by A viewed as a bimodule over itself.

Definition 2.15. The Hochschild complex of A with coefficients in an A-bimodule M is given by

$$C^*(A, M) = \mathbb{R}\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, M).$$

Clearly this is well-defined up to quasi-isomorphism. The Hochschild cohomology $HH^*(A, M)$ of A with coefficients in M is given by the cohomology of this complex.

We will be interested in the case $M = A$. In this case, the Hochschild cohomology of A encodes the deformation theory of A as a bimodule over itself. Note that

$$HH^0(A, A) = \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A) = Z(A).$$

For that reason, Hochschild cohomology is often called the "derived center" of A. In the literature, the Hochschild cochain complex of A is often implicitly taken to be $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(B_*(A), A) \cong \mathrm{Hom}_k(A^{\otimes *}, A)$ where $B_*(A)$ is the Bar resolution.

In 1962, M. Gerstenhaber noticed that Hochschild cohomology is equipped with a shifted Lie bracket $[-,-]_G$ and a commutative product $\smile$ such that $[f,-]_G$ is a derivation for the product. Such a structure is now called a Gerstenhaber algebra. We denote the operad governing Gerstenhaber algebras by $\mathcal{G}er$. In 1976, F. Cohen realzied that the Gerstenhaber operad is the homology of the topological operad of little 2-disks D_2. In particular, if C_* is an algebra over $C_*(D_2)$, then $H_*(C)$ is an algebra over $\mathcal{G}er$. This inspired P. Deligne conjecture in a 1993 letter that in fact the Hochschild cochain complex of an associative algebra A is an algebra over the chains on little 2-disks operad in such a way that the induced Gerstenhaber algebra structure on cohomology recovers Gerstenhaber's original one.

Multiple different proofs have been given that this is in fact true, see for example [Tam98], [Vor00], [MS02]. In particular, D. Tamarkin in [Tam98] constructed a map $\Psi_T : \mathcal{G}er_\infty \to Braces$ from the operad of homotopy Gerstenhaber algebras to the braces-operad, and also proved a formality result for the Gerstenhaber operad. Since $C^*(A, A)$ is canonically a braces-algebra, this solves Deligne's conjecture. Notably, Tamarkin's map Ψ_T depends on the choice of a Drinfeld associator.

In his letter, Deligne conjectured that the $C_*(D_2)$-algebra structure on the cochain level should come from the two multiplications on $\mathbb{R}\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A)$ induced by A being a monoidal unit. In particular, we have an inner multiplication coming from the bialgebra structure of A in the bimodule category. Let $\mathrm{End}(A) := \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A)$. Then we get

$$\begin{aligned} \mathrm{End}(A) \otimes_A \mathrm{End}(A) \otimes_A A &\xrightarrow{\cong} \mathrm{End}(A) \otimes_A \mathrm{End}(A) \otimes_A A \otimes_A A \\ &\cong \mathrm{End}(A) \otimes_A A \otimes_A \mathrm{End}(A) \otimes_A A \to A \otimes_A A \xrightarrow{\cong} A \end{aligned}$$

inducing a multiplication $\mathrm{End}(A) \otimes_A \mathrm{End}(A) \to \mathrm{End}(A)$. We also have an outer multiplication, the Yoneda product, induced by composition. This approach to proving the Deligne conjecture was examined in [HKV06], but is very difficult to link to existing definitions due to the incompatibility of the Boardman-Vogt tensor product with homotopical structure. In this paper, we will follow a similar approach to Hu-Kriz-Voronov, but we will use the language of ∞-operads. The main advantage of this is that we have access to a precise Dunn additivity statement 2.14. This makes it possible to get an actual $C_*(\mathbb{E}_2)$-algebra structure from two compatible $\mathbb{A}_\infty$-algebra structures, and to give precise meaning to the intuition that the Gerstenhaber bracket corresponds to the chain homotopy witnessing the compatibility of the two multiplications.

2.5 Polydifferential operators and formality morphisms

Let X be a separated scheme over k. Then the Hochschild cohomology of X should be a global version of the above definitions, such that if $X = \mathrm{Spec}(A)$ we recover definition 2.15. Historically, there have been multiple proposed definitions, for example by Grothendieck-Loday, Gerstenhaber-Schack and Swan. The perhaps most straight-forward generalization from the affine case was given by Swan, who defined the Hochschild cohomology of X to be

$$HH^*_S(X) = \mathrm{Ext}^*_{\mathcal{O}_{X\times_k X}}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X).$$

Unfortunately, this does not carry a Gerstenhaber bracket. For the case that X is smooth, Kontsevich gave an alternative definition, which does carry the structure of a Gerstenhaber algebra. He defined the complex of polydifferential operators on a regular algebra A to be the subcomplex

$$D_{\mathrm{poly}}(A) \subseteq C^*(A, A)$$

of maps $f : A^{\otimes n} \to A$ that are differential operators in each variable separately. These glue together to yield the sheaf of polydifferential operators on X

$$\mathcal{D}_{\mathrm{poly}}(X)(\mathrm{Spec}(A)) = D_{\mathrm{poly}}(A).$$

Then the Hochschild cohomology of X is the hypercohomology of the sheaf of polydifferential operators,

$$HH^*_K(X) = \mathbb{H}^*(X, \mathcal{D}_{\mathrm{poly}}).$$

Since $\mathcal{D}_{\mathrm{poly}}(X)$ is a sheaf of homotopy Gerstenhaber algebras by Tamarkin's solution of the Deligne conjecture, the hypercohomology inherits a Gerstenhaber algebra structure.

Associated to a smooth scheme X is also the sheaf of polyvector fields

$$\mathcal{T}_{\mathrm{poly}}(X) = \Lambda\mathcal{T}_X[-1],$$

where $\mathcal{T}_X$ is the tangent sheaf of X. This also carries the structure of a Gerstenhaber algebra with the bracket given by the Schouten-Nijenhuis bracket. In particular, its hypercohomology is also a Gerstenhaber algebra. The Hochschild-Kostant-Rosenberg theorem provides a canonical embedding

$$\mathcal{T}_{\mathrm{poly}}(X) \xrightarrow{\mathrm{HKR}} \mathcal{D}_{\mathrm{poly}}(X)$$

which is a quasi-isomorpism of complexes of sheaves. In case X is an affine space, Tamarkin's algebraic version of Kontsevich's formality theorem states that the HKR map lifts to a $\mathcal{G}er_\infty$-quasi-isomorpism, and in particular it induces an isomorphism of Gerstenhaber algebras on hypercohomology. For general X however, Calaque and Van den Bergh showed in [CVdB10] that the HKR map needs to be corrected by the square root of the Todd class of X to yield an isomorphism of Gerstenhaber algebras

$$\mathbb{H}^*(X, \mathcal{T}_{\mathrm{poly}}(X)) \xrightarrow{\mathrm{HKR}\circ(\mathrm{Td}(X)^{\frac{1}{2}}\wedge -)} HH^*_K(X).$$

3 The bracket operation on 2-algebras

3.1 The bracket operation of an $\mathbb{E}_2$-algebra

Consider the topological operad $\mathbb{E}_2$ of little 2-disks and its corresponding dg-operad $C_*(\mathbb{E}_2)$. If $C_*(\mathbb{E}_2) \to \mathrm{End}(A)$ is an algebra in the category of chain complexes over k, we in particular have an action of the 2-ary operation space

$$C_*(\mathbb{E}_2(2)) \otimes A^{\otimes 2} \to A,$$

and recalling that $\mathbb{E}_2(2) \simeq S^1$, taking homology yields a map

$$H_*(S^1) \otimes H_*(A)^{\otimes 2} \to H_*(A).$$

Since $H_*(S^1) \cong \mathbb{Z}[p] \oplus \mathbb{Z}[\gamma]$ for some choice of basepoint $p \in S^1$ and generating loop $\gamma : [0,1] \to S^1$, this yields two 2-ary operations on $H_*(A)$; one of degree zero induced by $[p]$

$$\smile : H_*(A) \otimes H_*(A) \to H_*(A)$$

and one of degree one induced by $[\gamma]$

$$[\cdot,\cdot] : H_*(A) \otimes H_*(A) \to H_*(A)[-1].$$

These two operations make $H_*(A)$ into a Gerstenhaber algebra. This discussion in particular shows that the bracket operation is induced by the chain level operation $A^{\otimes 2} \to A$ corresponding to any choice of generating loop γ of the homology of S^1. We can hence generalize this notion to a general symmetric monoidal ∞-category.

Definition 3.16. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let $A : \mathbb{E}_2^\otimes \to \mathcal{C}^\otimes$ be an $\mathbb{E}_2$-algebra in $\mathcal{C}^\otimes$. Then we call the image under A of $\gamma_t \in \mathrm{Map}_{\mathbb{E}_2^\otimes}(\langle 2\rangle, \langle 1\rangle)_1$ the bracket operation of A at m_t.

Let $A : \mathbb{E}_2^{\otimes} \to N_{\mathrm{dg}}(C^\circ)^{\otimes}$ be an $\mathbb{E}_2$-algebra in a symmetric monoidal dg model category. Then we get an induced map

$$\mathbb{E}_2(2) = \mathrm{Map}^{\alpha}_{\mathbb{E}_2^{\otimes}}(\langle 2\rangle, \langle 1\rangle) \to \mathrm{Map}_{N_{\mathrm{dg}}(C^\circ)^{\otimes}}(A(\langle 2\rangle), A(\langle 1\rangle)).$$

Let $A = A(\langle 1\rangle)$. Then we have a homotopy equivalence

$$\mathrm{Map}_{N_{\mathrm{dg}}(C^\circ)^{\otimes}}(A(\langle 2\rangle), A(\langle 1\rangle)) \simeq \mathrm{Map}_{N_{\mathrm{dg}}(C^\circ)}(A^{\otimes 2}, A).$$

Hence we get a map (well-defined up to homotopy)

$$\mathbb{E}_2(2) \to \mathrm{Map}_{N_{\mathrm{dg}}(C^\circ)}(A^{\otimes 2}, A) \simeq \mathrm{DK}\tau_{\geq 0}\,\mathrm{Map}_C(A^{\otimes 2}, A),$$

and therefore taking homology we get maps

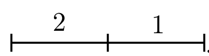
$$(\mathcal{G}er(2))_n \cong H_n(\mathbb{E}_2(2)) \to \mathrm{Hom}_{\mathcal{D}(C)}(A^{\otimes 2}, A[n]).$$

More generally, this procedure yields a Gerstenhaber algebra structure on A in the derived categor of C. In particular, we see that the bracket of this Gerstenhaber algebra is indeed given by the image of γ_0.

3.2 The bracket operation of a 2-algebra

If now $A : \mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes} \to \mathcal{C}^{\otimes}$ is a 2-algebra in $\mathcal{C}^{\otimes}$, the Dunn additivity theorem tells us that there exists an $\mathbb{E}_2$-algebra $\tilde{A} : \mathbb{E}_2^{\otimes} \to \mathcal{C}^{\otimes}$ such that the restriction of $\tilde{A}$ along $\rho : \mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes}$ is equivalent to A in the category of bifunctors. Fixing such an $\mathbb{E}_2^{\otimes}$-algebra, we can ask whether it is possible to express the bracket operations $\tilde{A}(\gamma_t)$ in terms of the original 2-algebra A.

Denote by $A \in \mathcal{C}$ the image $A(\langle 1\rangle, \langle 1\rangle)$, and let $\mu \in \mathrm{Hom}_{\mathbb{E}_1^{\otimes}}(\langle 2\rangle, \langle 1\rangle)_0$ be the element



We fix coCartesian lifts of inert maps in $\mathbb{E}_1^{\otimes}$ by using the correct enumerations of the full intervals. By abuse of notation, we denote those lifts by the inert map in Fin_* they lift. The key observation in expressing the bracket operations in terms of the orignal 2-algebra is given by the following theorem.

Theorem 3.17. *The images under A of the 2-simplices in $\mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes}$*

$$\begin{tikzcd}
(\langle 2\rangle, \langle 2\rangle) \arrow[r, "(\mathrm{id}_{\langle 2\rangle}, \mu)"] \arrow[d, "(\mu, \mathrm{id}_{\langle 2\rangle})"'] & (\langle 2\rangle, \langle 1\rangle) \arrow[d, "(\mu, \mathrm{id}_{\langle 1\rangle})"] \\
(\langle 1\rangle, \langle 2\rangle) \arrow[r, "(\mathrm{id}_{\langle 1\rangle}, \mu)"'] & (\langle 1\rangle, \langle 1\rangle)
\end{tikzcd}$$

$$\begin{tikzcd}
(\langle 1\rangle, \langle 2\rangle) \arrow[d, "(\mathrm{id}_{\langle 1\rangle}, (2,3))"'] \arrow[dr, "(\mathrm{id}_{\langle 1\rangle}, \mathrm{id}_{\langle 2\rangle})"] & \\
(\langle 1\rangle, \langle 4\rangle) \arrow[r, "(\mathrm{id}_{\langle 1\rangle}, (\mu, \mu))"'] & (\langle 1\rangle, \langle 2\rangle)
\end{tikzcd}$$

$$\begin{tikzcd}
(\langle 2\rangle,\langle 1\rangle) \arrow[r, "{((2,3),\mathrm{id}_{\langle 1\rangle})}"] \arrow[dr, "{(\tau,\mathrm{id}_{\langle 1\rangle})}"'] & (\langle 4\rangle,\langle 1\rangle) \arrow[d, "{((\mu,\mu),\mathrm{id}_{\langle 1\rangle})}"] \\
& (\langle 2\rangle,\langle 1\rangle)
\end{tikzcd}$$

induce a 2-simplex in $\mathcal{C}^{\otimes}$ up to homotopy

$$\begin{tikzcd}
(A,A) \arrow[dr, "\iota_{2,3}"] \arrow[drr, bend left, "id_{(A,A)}"] \arrow[ddr, bend right, "\tau"'] & & \\
& (A,A,A,A) \arrow[r, "{(m_2,m_2)}"] \arrow[d, "{(m_1,m_1)\circ\tau_{2,3}}"] & (A,A) \arrow[d, "m_1"] \\
& (A,A) \arrow[r, "m_2"'] & A
\end{tikzcd}$$

whose homotopy class is identified under the isomorphism $\tilde{A}\circ\rho \simeq A$ with the image under $\tilde{A}$ of the half twist between $t=0$ and $t=\pi$ in $\mathrm{Hom}_{\mathbb{E}_2^{\otimes}}(\langle 2\rangle,\langle 1\rangle)$.

Proof. First check that the 2-simplices in $\mathbb{E}_1^{\otimes}\times\mathbb{E}_1^{\otimes}$ indeed induce composable 2-simples in $\mathcal{C}^{\otimes}$ with the depicted boundaries. We immediately get diagrams

$$\begin{tikzcd}
& (A,A,A,A) \arrow[rr] \arrow[dd] & & (A,A) \arrow[dd] \\
A(\langle 2\rangle,\langle 2\rangle) \arrow[ur, "\simeq"] \arrow[rr] \arrow[dd] & & A(\langle 2\rangle,\langle 1\rangle) \arrow[ur, "\simeq"] \arrow[dd] & \\
& (A,A) \arrow[rr] & & A \\
A(\langle 1\rangle,\langle 2\rangle) \arrow[ur, "\simeq"] \arrow[rr] & & A(\langle 1\rangle,\langle 1\rangle) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A,A) \arrow[d] \arrow[dr] & \\
A(\langle 1\rangle,\langle 2\rangle) \arrow[ur, "\simeq"] \arrow[d] \arrow[dr] & (A,A,A,A) \arrow[r] & (A,A) \\
A(\langle 1\rangle,\langle 4\rangle) \arrow[r] \arrow[ur, "\simeq"] & A(\langle 1\rangle,\langle 2\rangle) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A,A) \arrow[r] \arrow[dr] & (A,A,A,A) \arrow[d] \\
A(\langle 2\rangle,\langle 1\rangle) \arrow[ur, "\simeq"] \arrow[r] \arrow[dr] & A(\langle 4\rangle,\langle 1\rangle) \arrow[ur, "\simeq"] \arrow[d] & (A,A) \\
& A(\langle 2\rangle,\langle 1\rangle) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

so it suffices to show that these fit together into the depicted 2-simplex. To this end we check that the maps $(A, A, A, A) \to (A, A)$ in the square agree with the respective maps $(A, A, A, A) \to (A, A)$ in the triangles, and similarly for the two maps $(A, A) \to (A, A, A, A)$ in the different triangles. To this end note that it suffices to show that each of those pairs of maps agrees after postcomposition with coCartesian lifts of the ρ^i. Consider first the maps induced by $(\langle 2\rangle, \langle 2\rangle) \xrightarrow{\mathrm{id}_{\langle 2\rangle}, \mu} (\langle 2\rangle, \langle 1\rangle)$ and $(\langle 1\rangle, \langle 4\rangle) \xrightarrow{\mathrm{id}_{\langle 1\rangle}, (\mu,\mu)} (\langle 1\rangle, \langle 2\rangle)$. We have a factorization

$$\begin{tikzcd}
(\langle 2\rangle, \langle 2\rangle) \arrow[r, "{\mathrm{id},\mu}"] \arrow[dr, "{\rho^i,\mathrm{id}}"'] & (\langle 2\rangle, \langle 1\rangle) \arrow[r, "{\rho^i,\mathrm{id}}"] & (\langle 1\rangle, \langle 1\rangle) \\
& (\langle 1\rangle, \langle 2\rangle) \arrow[ur, "{\mathrm{id},\mu}"'] &
\end{tikzcd}$$

in $\mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes}$, where the lower lhs map lies over the inert map $1, 2, *, * : \langle 4\rangle \to \langle 2\rangle$ if $i = 1$ and over the inert map $*, *, 1, 2$ if $i = 2$. Since inert pairs are sent to inert maps by A, this diagram maps to

$$\begin{tikzcd}
(A, A, A, A) \arrow[r, "{A(\mathrm{id},\mu)}"] \arrow[dr] & (A, A) \arrow[r, "\rho^i"] & A \\
& (A, A) \arrow[ur, "m_2"'] &
\end{tikzcd}.$$

where $(A, A, A, A) \to (A, A)$ is an inert lift. Now on the other hand, we also have a factorization

$$\begin{tikzcd}
(\langle 1\rangle, \langle 4\rangle) \arrow[r, "{\mathrm{id},(\mu,\mu)}"] \arrow[dr, "{\mathrm{id},f_i}"'] & (\langle 1\rangle, \langle 2\rangle) \arrow[r, "{\mathrm{id},\rho^i}"] & (\langle 1\rangle, \langle 1\rangle) \\
& (\langle 1\rangle, \langle 2\rangle) \arrow[ur, "{\mathrm{id},\mu}"'] &
\end{tikzcd}$$

where f_i is $1, 2, *, *$ if $i = 1$ and $*, *, 1, 2$ if $i = 2$. This again is sent by A to

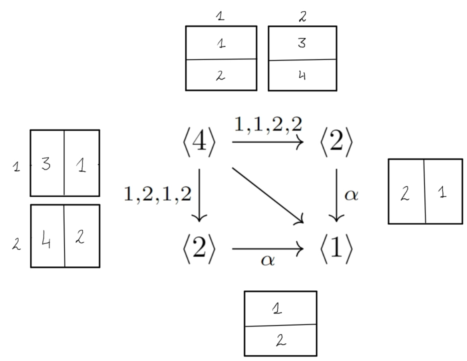
$$\begin{tikzcd}
(A, A, A, A) \arrow[r, "{A(\mathrm{id},(\mu,\mu))}"] \arrow[dr] & (A, A) \arrow[r, "\rho^i"] & A \\
& (A, A) \arrow[ur, "m_2"'] &
\end{tikzcd},$$

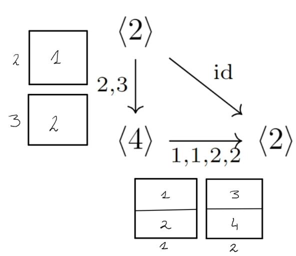
showing that our two maps $(A, A, A, A) \to (A, A)$ indeed agree up to homotopy. An analogous analysis can be carried out with the other two maps $(A, A, A, A) \to (A, A)$. For the two inclusions $(\langle 1\rangle, \langle 2\rangle) \to (\langle 1\rangle, \langle 4\rangle)$ and $(\langle 2\rangle, \langle 1\rangle) \to (\langle 4\rangle, \langle 1\rangle)$, it suffices to show that the two unit maps coming from $(\langle 1\rangle, \langle 0\rangle) \to (\langle 1\rangle, \langle 1\rangle)$ and $(\langle 0\rangle, \langle 1\rangle) \to (\langle 1\rangle, \langle 1\rangle)$ agree. To this end, note that both $(\langle 0\rangle, \langle 0\rangle) \to (\langle 1\rangle, \langle 0\rangle)$ and $(\langle 0\rangle, \langle 0\rangle) \to (\langle 0\rangle, \langle 1\rangle)$ are sent to the identity on the empty tuple by A, up to homotopy, since $\mathcal{C}^{\otimes}_{\langle 0\rangle}$ is contractible. Then the image of the diagram

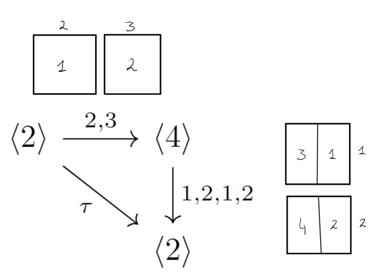
$$
\begin{tikzcd}
 & (\langle 1\rangle,\langle 0\rangle) \arrow[dr] & \\
(\langle 0\rangle,\langle 0\rangle) \arrow[ur] \arrow[rr] \arrow[dr] & & (\langle 1\rangle,\langle 1\rangle) \\
 & (\langle 0\rangle,\langle 1\rangle) \arrow[ur] &
\end{tikzcd}
$$

under A shows that the two unit maps are homotopic.

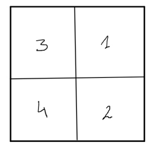
We now show that the depicted 2-simplex in $\mathcal{C}^{\otimes}$ indeed corresponds to the image of the half twist under $\tilde{A}$. To this end, examine the images of the three 2-simplices in $\mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes}$ under $\rho : \mathbb{E}_1^{\otimes} \times \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes}$. We get





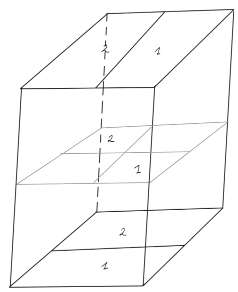


Note that the paths forming the fillings of the two triangles in the square diagram are both the constant path at



,

while the fillings of the triangle diagrams are given by continuously enlarging the respective rectangles. Composing those in the simplicial category $\mathbb{E}_2^{\Delta,\otimes}$ we hence get the half twist



.

Applying $\tilde{A}$ to the above simplices induces a diagram of 2-simplices in $\mathcal{C}^{\otimes}$

$$\begin{tikzcd}
(\tilde{A},\tilde{A}) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (\tilde{A},\tilde{A},\tilde{A},\tilde{A}) \arrow[r] \arrow[d] & (\tilde{A},\tilde{A}) \arrow[d] \\
& (\tilde{A},\tilde{A}) \arrow[r] & \tilde{A}
\end{tikzcd}$$

and by construction the isomorphism between $\tilde{A} \circ \rho$ and A identifies this diagram with the one in the statement. Since maps between ∞-categories respect composition, this proves the claim. $\square$

We can repeat the analysis in this theorem for the other parts of the classical Eckmann-Hilton argument. In particular, we get representations of the image under $\tilde{A}$ of all four different parts of the double twist in terms of compositions of 2-simplices in the image of A. As a corollary, we obtain

Corollary 3.18. *The homotopy class of the bracket on $\tilde{A}$ at $t = 0$ can be produced as a composition of the following 2-simplices in $\mathcal{C}^{\otimes}$*

$$\begin{tikzcd}
(A,A) \arrow[dr, "\iota_{2,3}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\tau"'] & & \\
& (A,A,A,A) \arrow[r, "{(m_2,m_2)}"] \arrow[d, "{(m_1,m_1)\circ\tau_{2,3}}"] & (A,A) \arrow[d, "m_1"] \\
& (A,A) \arrow[r, "m_2"'] & A
\end{tikzcd},$$

$$\begin{tikzcd}
(A,A) \arrow[dr, "\iota_{1,4}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& (A,A,A,A) \arrow[r, "{(m_2^{\mathrm{op}},m_2^{\mathrm{op}})}"] \arrow[d, "{(m_1^{\mathrm{op}},m_1^{\mathrm{op}})\circ\tau_{2,3}}"] & (A,A) \arrow[d, "m_1^{\mathrm{op}}"] \\
& (A,A) \arrow[r, "m_2^{\mathrm{op}}"'] & A
\end{tikzcd},$$

$$\begin{tikzcd}
(A,A) \arrow[dr, "\iota_{2,3}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\tau"'] & & \\
& (A,A,A,A) \arrow[r, "{(m_2^{\mathrm{op}},m_2^{\mathrm{op}})}"] \arrow[d, "{(m_1^{\mathrm{op}},m_1^{\mathrm{op}}))\circ\tau_{2,3}}"] & (A,A) \arrow[d, "m_1^{\mathrm{op}}"] \\
& (A,A) \arrow[r, "m_2^{\mathrm{op}}"'] & A
\end{tikzcd},$$

$$\begin{tikzcd}
(A,A) \arrow[dr, "\iota_{1,4}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& (A,A,A,A) \arrow[r, "{(m_2,m_2)}"] \arrow[d, "{(m_1,m_1)\circ\tau_{2,3}}"] & (A,A) \arrow[d, "m_1"] \\
& (A,A) \arrow[r, "m_2"'] & A
\end{tikzcd}.$$

Corollary 3.19. *If $\mathcal{C}^{\otimes}$ is the dg nerve of a symmetric monoidal dg model category C, then the bracket at m_0 on $\tilde{A}$ is given by the chain homotopy*

$$h\iota_{2,3} + h^{op}\iota_{1,4} + h^{op}\iota_{2,3} + h\iota_{1,4}$$

up to higher chain homotopies. Here h and h^{op} are the chain homotopies corresponding to the image of the square diagram in $\mathbb{E}_1 \times \mathbb{E}_1$ for the multiplications and their opposite multiplications respectively.

Proof. Note first that by construction of the symmetric monoidal structure on $\mathcal{C}$, a morphism $(A, \dots, A) \to (A, \dots, A)$ in $\mathcal{C}^{\otimes}$ corresponds to a map $A^{\otimes m} \to A^{\otimes n}$ in C which is unique up to chain homotopy. Similarly, a 2-simplex between such maps corresponds to a chain homotopy between the corresponding maps in C. Fix such maps in C corresponding to all the involved diagrams. Horizontal composition of maps is strictly defined in dg categories, and hence we have well defined whiskering compositions $h\iota_{2,3}$, $h\iota_{1,4}$, $h^{\mathrm{op}}\iota_{2,3}$ and $h^{\mathrm{op}}\iota_{1,4}$. Finally, note that horizontal composition of chain homotopies is given by addition. □

4 Recovering the Hochschild complex as $\mathbb{E}_1$-center

Often the Hochschild cohomology of a k-algebra is called its "derived center". In this section we will show that this statement is true in a very precise sense. Namely, the Hochschild complex is the $\mathbb{E}_1$-center in the derived ∞-category of chain complexes.

4.1 The dg nerve

Theorem 4.20. *Let $\mathcal{C}$ be a dg category that is tensored over the category of chain complexes of k-modules, let $\mathcal{C}'$ be a full dg subcategory, and let W be a collection of morphisms in $\mathcal{C}'$ that are isomorphisms in the homotopy category. Assume that the following conditions are satisfied:*

- *Every isomorphism in $\mathcal{C}'$ belongs to W.*
- *The set W satisfies the 2-out-of-3 property.*
- *For all $X \in \mathcal{C}'$, we also have $N_*(\Delta^1) \otimes X \in \mathcal{C}'$.*
- *For each $X \in \mathcal{C}'$, the map $N_*(\Delta^1) \otimes X \to X$ induced by the map $[1] \to [0]$ belongs to W.*

Then the canonical map $\theta : N(\mathcal{C}'_0) \to N_{dg}(\mathcal{C}')$ induces an equivalence of ∞-categories $\theta' : N(\mathcal{C}'_0)[W^{-1}] \simeq N_{dg}(\mathcal{C}')$.

Proof. The above conditions are exactly what is needed to repeat the proof of [Lur17, 1.3.4.5] replacing $\mathrm{Ch}(\mathcal{A})$ with $\mathcal{C}$ and $\mathrm{Ch}(\mathcal{A})'$ with $\mathcal{C}'$. □

Corollary 4.21. *Let $\mathcal{C}$ be a dg model category, and let $\mathcal{C}^\circ$ be the full subcategories on bifibrant objects. Then the map $N(\mathcal{C}_0^c) \to N_{dg}(\mathcal{C}^\circ)$ exhibits the dg nerve as the underlying ∞-category of $\mathcal{C}_0$.*

Proof. Let $\mathcal{C}_\Delta$ be the simplicial category obtained from $\mathcal{C}$. Since $N(\mathcal{C}_0^c)[W^{-1}] \simeq N(\mathcal{C}^\circ)[W^{-1}]$ is suffices to take $\mathcal{C}' = \mathcal{C}^\circ$ above and W the set of homotopy equivalences. Between bifibrant objects, left homotopy agrees with chain homotopy in the sense a map $h : N(\Delta^1) \otimes X \to Y$ with the correct restrictions to $\{0\}$ and $\{1\}$. In particular,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_0}(N_*(\Delta^1) \otimes X, Y) &\cong \mathrm{Hom}_{\mathrm{Ch}(k)}(N_*(\Delta^1), \mathrm{Map}_{\mathcal{C}}(X,Y)) \\
&\cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}}(N_*(\Delta^1), \tau_{\geq 0} \mathrm{Map}_{\mathcal{C}}(X,Y) \\
&\cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^1, \mathrm{DK}_\bullet \tau_{\geq 0} \mathrm{Map}_{\mathcal{C}}(X,Y)) \\
&\cong \mathrm{Map}_{\mathcal{C}_\Delta}(X,Y)_1
\end{aligned}$$

so chain homotopies correspond to 1-chains of the mapping complex of $\mathcal{C}$. One checks directly that the diagram making $h : N_*(\Delta^1) \otimes X \to Y$ into a homotopy between $f, g \in \mathrm{Hom}_{\mathcal{C}_0}(X,Y)$ forces the corresponding 1-chain $z \in \mathrm{Map}_{\mathcal{C}}(X,Y)_1$ to satisfy $dz = f - g$. This shows that homotopy equivalences in $\mathcal{C}_0^\circ$ become isomorphisms in the homotopy category $h\mathcal{C}_\Delta$, which is isomorpic to the homotopy category $h\mathcal{C}$. Clearly every isomorphism is a homotopy equivalence and the set of homotopy equivalences satisfies 2-out-of-3. Since $\mathcal{C}$ is tensored over $\mathrm{Ch}(k)$, the map $\otimes : \mathrm{Ch}(k)\times\mathcal{C} \to \mathcal{C}$ is a left Quillen bifunctor. The complex $N_*(\Delta^1)$ is bounded below and therefore cofibrant, and hence $N_*(\Delta^1) \otimes -$ preserves cofibrant objects. At the same time, $\mathcal{C}$ is also powered over $\mathrm{Ch}\, k$ and hence the functor $N_*(\Delta^1) \otimes -$ is right adjoint to the left Quillen bifunctor $N_*(\Delta^1)^\vee \otimes -$. In particular, it preserves fibrant objects. This shows that $N_*(\Delta^1) \otimes X \in \mathcal{C}^\circ$. Finally, note that $d_0 : k \to N_*(\Delta^1)$ is a trivial cofibration in $\mathrm{Ch}(k)$, and thus if $X \in \mathcal{C}^\circ$, the map $X \cong k \otimes X \to N_*(\Delta^1) \otimes X$ is again a trivial cofibration. Now the map $N_*(\Delta^1) \to k$ is a left inverse to d_0 and in particular

$$X \to N_*(\Delta^1) \otimes X \to X$$

is the identity on X and thus a weak equivalence. By 2-out-of-3, this means that $N_*(\Delta^1) \otimes X \to X$ must be a weak equivalence. □

Remark 4.22. Note that for any dg category $\mathcal{C}$, we have an equivalence of ∞-categories $N_{\mathrm{hc}}(\mathcal{C}_\Delta) \to N_{\mathrm{dg}}(\mathcal{C})$. For simplicial model categories, the homotopy coherent nerve of the bifibrant objects is always the ∞-category underlying the model category, but $\mathcal{C}_\Delta$ is not tensored and cotensored over sSet and therefore does not satisfy the requirements of this theorem. The above corollary then shows that we get this relationship between the homotopy coherent nerve and the model category regardless.

If $\mathcal{C}$ is a (symmetric) monoidal model category, then [Lur17, 4.1.7.6] shows that $N(\mathcal{C}^c)[W^{-1}]$ is a (symmetric) monoidal ∞-category. If $\mathcal{C}$ is also a simplicial model category and the (symmetric) monoidal structure is compatible with the simplicial enrichment, then [Lur17, 4.1.7.16] shows that the (symmetric) monoidal structure on this ∞-category is given by $N_{\mathrm{hc}}((\mathcal{C}^\circ)^\otimes)$, and in fact one readily checks that the same hold if $\mathcal{C}$ is just weakly simplicial in the same sense as above. We would like to use this to argue that the dg nerve of a (symmetric) monoidal dg model category $\mathcal{C}$ is a (symmetric) monoidal ∞-category and also presents the (symmetric) monoidal structure of $N(\mathcal{C}^c)[W^{-1}]$, but unfortunately the Dold-Kan functor is not symmetric, and thus does not send operads to operads. Nevertheless, it is homotopy symmetric lax monoidal, and in fact V. Hinich proved

Proposition 4.23 ([Hin13], 3.2.3)**.** *The dg nerve* $N_{\mathrm{dg}} : N(\mathrm{Cat}_{\mathrm{dg}})[W_{\mathrm{dg}}^{-1}] \to N(\mathrm{Cat}_\Delta)[W_\Delta^{-1}] \simeq \mathrm{Cat}_\infty$ *from the symmetric monoidal ∞-category of dg-categories to the symmetric monoidal ∞-category of ∞-categories is lax symmetric monoidal. In particular, it is a morphism of ∞-operads. It thus induces a map from the ∞-category of symmetric monoidal dg-categories to the ∞-category of symmetric monoidal ∞-categories.*

Corollary 4.24. *The dg nerve induces a map* $\mathrm{Alg}_{\mathcal{LM}^\otimes}(N(\mathrm{Cat}_{\mathrm{dg}})[W_{\mathrm{dg}}^{-1}]) \to \mathrm{Alg}_{\mathcal{LM}^\otimes}(\mathrm{Cat}_\infty)$*. By 4.25, this means that a dg category left tensored over a monoidal dg category yields an ∞-category left tensored over a monoidal ∞-category.*

4.2 Rectification of algebras over an ∞-operad

Let $\mathcal{O}$ be a topological operad. Then we get a dg operad $C_*(\mathcal{O}; k)$ by applying the singular chains functor with coefficients in our field k. We want to view algebras in a symmetric monoidal dg model category $\mathcal{C}$ over this dg operad as algebras over the ∞-operad $N^\otimes(\mathrm{Sing}_\bullet\mathcal{O})$ in the symmetric monoidal ∞-category $N_{\mathrm{dg}}(\mathcal{C}^\circ)$. We generalize the rectification results of V. Hinich [Hin13] and of D. Pavlov and J. Scholbach [PS14].

Let $\mathcal{C}$ be a symmetric monoidal model category that is enriched in the projective model category of chain complexes over k. Suppose further that $\mathcal{C}$ is cofibrantly generated and symetrically flat, and that $C_*(\mathcal{O};k)$ is admissible and well-pointed in $\mathcal{C}$, and that it admits a lax monoidal fibrant replacement functor. The construction in [Hin13, 4.2] generalizes directly to give a functor

$$\phi : N(\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})^c) \to \mathrm{Alg}_{N^\otimes(\mathcal{O})}(N_{\mathrm{dg}}(\mathcal{C}^\circ))$$

that carries weak equivalences to equivalences, and therefore yields a comparison map

$$\Phi : N(\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})^c)[W^{-1}_{\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})}] \to \mathrm{Alg}_{N^\otimes(\mathcal{O})}(N_{\mathrm{dg}}(\mathcal{C}^\circ)).$$

We want to show that this functor is an equivalence of ∞-categories.

Theorem 4.25. *Let $\mathcal{O}$ and $\mathcal{C}$ as above. Then Φ is an equivalence of ∞-categories.*

Proof. We use corollary 4.7.3.16 in [Lur17]. Following the reasoning in [PS14, 7.11], we may assume that $C_*(\mathcal{O};k)$ is Σ-cofibrant in $\mathcal{C}$. Now consider

$$\begin{tikzcd}
N(\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})^c)[W^{-1}_{\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})}] \arrow[rr, "\Phi"] \arrow[dr, "G"'] & & \mathrm{Alg}_{N^\otimes(\mathcal{O})}(N(\mathcal{C}^c)[W^{-1}]) \arrow[dl, "G'"] \\
& (N(\mathcal{C}^c)[W^{-1}])^{[\mathcal{O}]} &
\end{tikzcd}$$

Steps (a)-(c) can be proven exactly like in [Lur17, 4.5.4.7], by just replacing the commutative operad by $\mathcal{O}$. For step (d), it is clear that G is conservative since the weak equivalences in $\mathrm{Alg}_{C_*(\mathcal{O};k)}(\mathcal{C})$ are transferred from the ones in $\mathcal{C}$ via the forgetful functor. To show that G preserves geometric realization of simplicial objects, it suffices to show that it preserves homotopy sifted colimits. This is shown in [PS14, prop. 7.9]. Finally, we need to show that the canonical transformation $G' \circ F' \to G \circ F$ is an equivalence, where F and F' are the left adjoints of G and G' respectively. This boils down to showing that for any cofibrant object $X \in \mathcal{C}^{[\mathcal{O}]}$, the strict free $C_*(\mathcal{O};k)$-algebra generated by X is also a free $N^\otimes(\mathcal{O})$-algebra in the sense of [Lur17, def. 3.1.3.1]. In [Hin13, 4.3.4], Hinich has described an analog of [Lur17, 3.1.3.13] for free algebras generated by objects of different colors. The arguments described there go through if we replace the category of chain complexes by $\mathcal{C}$. This finishes the argument. ∎

Proposition 4.26. *Let $A : \mathbb{E}_2^\otimes \to N_{\mathrm{dg}}(C^\circ)^\otimes$ be an $\mathbb{E}_2$-algebra in a symmetric monoidal dg model category. Let A^{str} be a homotopy preimage of A under Φ. Without loss of generality, assume that the underlying object of A^{str} is fibrant. Then there is a chain homotopy*

$$\begin{tikzcd}
A^{\mathrm{str}\otimes 2} \arrow[r, "A^{\mathrm{str}}(m_0)"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\Downarrow"] & A^{\mathrm{str}} \arrow[d, "\simeq"] \\
A^{\otimes 2} \arrow[r, "A(m_0)"'] & A
\end{tikzcd}$$

and a chain homotopy of chain homotopies

$$\begin{tikzcd}
A^{\mathrm{str}\otimes 2} \arrow[r, bend left, "A^{\mathrm{str}}(m_0)"] \arrow[r, bend right, "A^{\mathrm{str}}(m_0)"'] \arrow[r, phantom, "\Downarrow A^{\mathrm{str}}(\gamma_0)"] \arrow[d, "\simeq"'] & A^{\mathrm{str}} \arrow[d, "\simeq"] \\
A^{\otimes 2} \arrow[r, bend left, "A(m_0)"] \arrow[r, bend right, "A(m_0)"'] \arrow[r, phantom, "\Downarrow A(\gamma_0)"] & A
\end{tikzcd}.$$

In particular, the induced Gerstenhaber algebra in the derived category agrees with the one constructed directly from A in section 3.1.

Proof. There is a natural isomorphism $\eta : \Delta^1 \times \mathbb{E}_2^{\otimes} \to N_{\mathrm{dg}}(C^{\circ})^{\otimes}$ between $\Phi(A^{\mathrm{str}})$ and A. In particular, we get isomorphisms $\Phi(A^{\mathrm{str}})(\langle n\rangle) \to A(\langle n\rangle)$, that correspond to isomorphisms $\Phi(A^{\mathrm{str}})^{\otimes n} \to A^{\otimes n}$. All the (higher) chain homotopies in the statement now correspond to the evaluation of η at the appropriate simplices in $\Delta^1 \times \mathbb{E}_2^{\otimes}$. For example, the 1-simplex (e, m_0) yields a map $\Phi(A^{\mathrm{str}})^{\otimes 2} \to A$ in C, and the 2-simplex $(s_0 e, s_1 m_0)$ yields a chain homotopy making $\eta(e, m_0)$ into a composition of $\Phi(A^{\mathrm{str}})(m_0)$ and $\eta(e, s_0\langle 1\rangle)$. $\square$

4.3 The Hochschild complex

In this section, let k be a field with $k \supseteq \mathbb{Q}$. Let A be an associative k-algebra. Then Hochschild cohomology of A is given by the Ext-groups of A as a bimodule over itself, i.e.

$$\mathrm{HH}^*(A, A) = \mathrm{Ext}^*_{A\otimes_k A^{\mathrm{op}}}(A, A).$$

In particular, this is computed by the cochain complex

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_*, A)$$

for any projective resolution P_* of A as a A-bimodule. Classically, one takes P_* to be the Bar-resolution. Note that the cochain complex above is the non-negatively graded cochain complex associated to the non-positively graded chain complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, A_*)$ where we view A as chain complex concentrated in degree zero. As such we have a quasi-isomorphism $\underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, A_*) \simeq \underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, P_*)$.

We want to use the equivalence constructed above to view A as an $\mathbb{E}_1$-algebra in the symmetric monoidal ∞-category $\mathcal{C}^{\otimes} := N_{\mathrm{dg}}(\mathrm{Ch}(k))$. To this end, let $\phi : C_*(\mathbb{E}_1) \xrightarrow{\simeq} \mathrm{Assoc}$ is the projection map, then we get $\phi^* A \in \mathrm{Alg}_{C_*(\mathbb{E}_1)}(\mathrm{Ch}(k))$. If $\tilde{A} \xrightarrow{\simeq} \phi^* A$ is a cofibrant replacement, we can then use theorem 4.25 to get an object $\tilde{A} \in \mathrm{Alg}_{\mathbb{E}_1}(N_{\mathrm{dg}}(\mathrm{Ch}(k))$. The rest of this section will go into proving the following theorem.

Theorem 4.27. *For any projective resolution P_* of A as an A^e-module, the evaluation map*

$$\underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, P_*) \otimes_k P_* \to P_*$$

makes Hochschild complex of A into a center of $\tilde{A} \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Ch}(k))$. In particular, this makes $\underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, P_*)$ into an element of $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(N_{\mathrm{dg}}(\mathrm{Ch}(k)))) \simeq \mathrm{Alg}_{\mathbb{E}_2}(N_{\mathrm{dg}}(\mathrm{Ch}(k)))$.

Corollary 4.28. *The strictification of the Hochschild complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}(A^e)}(P_*, P_*) \in \mathrm{Ch}(k)$ naturally carries the structure of a $C_*(\mathbb{E}_2)$-algebra. This $C_*(\mathbb{E}_2)$-algebra structure recovers the classical Gerstenhaber algebra structure in homology.*

Sketch! The first part follows directly from the rectification theorem 4.25. For the second part, we use corollary 3.19. It hence suffices to show that the image of m_0 and γ_0 under the $\mathbb{E}_2$-algebra structure above are chain homotopic to the cup product and bracket respectively. This follows from theorem 4.32. □

4.4 The center as endomorphism object of bimodules

To show that this indeed yields a center for $\tilde{A}$, we will employ some methods to compute the center in algebra categories. Throughout this chapter, let $\mathcal{O}^\otimes$ be a coherent ∞-operad; we will mostly be interested in the case $\mathcal{O}^\otimes = \mathbb{E}_1^\otimes$. Let $\mathcal{C}^\otimes \to \mathit{Fin}_*$ be a symmetric monoidal ∞-category and consider the unique bifunctor of ∞-operads $\mathcal{O}^\otimes \times \mathcal{LM}^\otimes \to \mathit{Fin}_* \times \mathit{Fin}_* \xrightarrow{\wedge} \mathit{Fin}_*$. We get a coCartesian fibration $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^\otimes \to \mathcal{LM}^\otimes$ with fibers over $\mathfrak{a}$ and $\mathfrak{m}$ respectively both equivalent to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$. This comes from the fact that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits the structure of a symmetric monoidal ∞-category and is hence left tensored over itself. If A admits a center, it has the structure of an associative algebra object in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{a}}$. In particular, if $\mathcal{O}^\otimes = \mathbb{E}_k^\otimes$ is a little k-cubes operad, by Dunn additivity we get

$$\mathfrak{z}(A) \in \mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{E}_{k+1}^\otimes}(\mathcal{C}).$$

Let A be an associative algebra over k. Then there is an equivalence of categories between algebra objects in the monoidal category of A-bimodules and associative algebras under A. In particular, an A-bimodule structure on an associative algebra B is the same as an algebra morphism $A \to B$, and the equivalence is given by sending the bimodule structure to its unit morphism. Now the centralizer of an algebra morphism $f : A \to B$ is defined as

$$\mathfrak{z}(f) = \{b \in B : \forall a \in A : f(a)b = bf(a)\}.$$

Viewing the data of f as an A-bimodule structure on B, we can see that this agrees with the set of A-bimodule maps from A to B:

$$\mathfrak{z}(f) \cong \mathrm{Hom}_{\mathrm{Bimod}_A}(A, B).$$

To generalize this relationship between the center and bimodule morphisms to ∞-operads, one needs to first recover the statement on algebra objects in the monoidal category of bimodules. To this end, let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$ and let $\overline{A} \in \mathrm{LMod}_{\mathbb{1}}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}})$ be a lift of A as a module over the trivial algebra. We get a coCartesian $\mathcal{LM}^\otimes$-family of $\mathcal{O}$-operads

$$\mathcal{C}^\otimes \times_{\mathit{Fin}_*} (\mathcal{O}^\otimes \times \mathcal{LM}^\otimes) \to \mathcal{O}^\otimes \times \mathcal{LM}^\otimes,$$

and we can regard $\overline{A}$ as a coCartesian $\mathcal{LM}^\otimes$-family of $\mathcal{O}$-algebras by noting that there is a bijection

$$\mathrm{Fun}_{\mathcal{LM}^\otimes}(\mathcal{LM}^\otimes, \mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^\otimes}(\mathcal{C}^\otimes \times_{\mathit{Fin}_*} (\mathcal{O}^\otimes \times \mathcal{LM}^\otimes))) \simeq \mathrm{Fun}_{\mathcal{LM}^\otimes}(\mathcal{LM}^\otimes, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^\otimes).$$

We can view A as an $\mathcal{O}$-module over itself, and hence $\overline{A}$ also determines a coCartesian $\mathcal{LM}^{\otimes}$-family of $\mathcal{O}$-algebras in the coCartesian $\mathcal{LM}^{\otimes}$-family of $\mathcal{O}$-operads

$$\overline{C}^{\otimes} := \mathrm{Mod}_{\overline{A}}^{\mathcal{O},\mathcal{LM}^{\otimes}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes}))^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes}.$$

This notation allows us to identify algebra objects in the category of A-bimodules as algebras under A.

Proposition 4.29 (Proposition 5.3.1.27 [Lur17])**.** *The forgetful functor*

$$\theta : \mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\mathrm{Mod}_{\overline{A}}^{\mathcal{O},\mathcal{LM}^{\otimes}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes}))^{\overline{A}_{\mathcal{LM}^{\otimes}/}} \to \mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes}))^{\overline{A}_{\mathcal{LM}^{\otimes}/}}.$$

Note that for all $s \in \mathcal{LM}^{\otimes}$, the algebra $\overline{A}_s \in \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_{\overline{A}_s}^{\mathcal{O}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \{s\})))$ is the trivial algebra, so in particular for $s = \mathfrak{m}$ we get an equivalence

$$\theta_{\mathfrak{m}} : \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_{A}^{\mathcal{O}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \{\mathfrak{m}\}))) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \{\mathfrak{m}\}))^{A/} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}^{A/}.$$

Since $\overline{A}(\mathfrak{a})$ is the trivial algebra, we have an equivalence $\overline{\mathcal{C}}_{\mathfrak{a}}^{\otimes} \simeq \mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \{\mathfrak{a}\})$, and therefore $\mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{a}} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{a}}$. To find the centralizer of id_A in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$, it hence suffices to find the centralizer of id_A in $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_{A}^{\mathcal{O}}(\mathcal{C}^{\otimes} \times_{Fin_*} (\mathcal{O}^{\otimes} \times \{\mathfrak{m}\})))$, in which A is the trivial algebra. After that, we can use proposition 2.13 to get the center $\mathfrak{z}(A) \in \mathrm{Alg}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{a}})$ of A.

Theorem 4.30 (Proposition 5.3.1.29 [Lur17])**.** *Suppose that for all $X \in \mathcal{O}$, there exists a morphism object $\mathrm{Mor}_{\overline{\mathcal{C}}_{X,\mathfrak{m}}}(A(X), A(X)) \in \overline{\mathcal{C}}_{X,\mathfrak{a}}$. Then there exists a centralizer $\mathfrak{z}(\mathrm{id}_A) \in \mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{a}}$. Furthermore, if $Z \in \mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{a}}$, then a commutative diagram*

$$\begin{tikzcd} & Z \otimes A \arrow[dr] & \\ A \arrow[ur] \arrow[rr, "\mathrm{id}_A"'] & & A \end{tikzcd}$$

exhibits Z as the centralizer of id_A if and only if for all $X \in \mathcal{O}$, the induced map $Z(X) \otimes A(X) \to A(X)$ exhibits $Z(X)$ as a morphism object of $A(X)$ and $A(X)$.

Proof. By definition, the centralizer is a final object of the ∞-category

$$\mathcal{A} := (\mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{a}})_{\mathbb{1}} \times_{(\mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{m}})_{A/}} (\mathrm{Alg}_{/\mathcal{O}}^{\mathcal{LM}^{\otimes}}(\overline{\mathcal{C}})_{\mathfrak{m}})_{A//A}.$$

Since $\overline{A}_s$ is the trivial algebra in $\overline{\mathcal{C}}_s^{\otimes}$ for $s \in \{\mathfrak{a}, \mathfrak{m}\}$, we can use [Lur17, Theorem 2.2.2.4] to get an $\mathcal{O}^{\otimes}$-monoidal ∞-category

$$\mathcal{E}^{\otimes} := (\overline{\mathcal{C}}_{\mathfrak{a}}^{\otimes})_{\mathbb{1}_{\mathcal{O}}/} \times_{(\overline{\mathcal{C}}_{\mathfrak{m}}^{\otimes})_{A_{\mathcal{O}}/}} (\overline{\mathcal{C}}_{\mathfrak{m}}^{\otimes})_{A_{\mathcal{O}}//A_{\mathcal{O}}} \to \mathcal{O}^{\otimes}$$

such that $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{E}) \simeq \mathcal{A}$. Finally, use that limits in algebra categories are computed object-wise by [Lur17, 3.2.2.5], and hence we are reduced to showing that for each $X \in \mathcal{O}$, the fiber $\mathcal{E}_X$ admits a final object. But a final object in

$$\mathcal{E}_X \simeq (\overline{\mathcal{C}}_{X,\mathfrak{a}})_{\mathbb{1}(X)/} \times_{(\overline{\mathcal{C}}_{X,\mathfrak{m}})_{A(X)/}} (\overline{\mathcal{C}}_{X,\mathfrak{m}})_{A(X)//A(X)}$$

is equivalent to a morphism object $\mathrm{Mor}_{\overline{\mathcal{C}}_{X,\mathfrak{m}}}(A(X), A(X))$ by proposition 2.4. $\square$

Corollary 4.31. *Let $\mathcal{O}^{\otimes} = \mathbb{E}_1^{\otimes}$, and let $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ be an $\mathbb{E}_1$-algebra in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. Assume that the morphism object $\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C}^{\otimes} \times_{Fin_*} \mathbb{E}_1^{\otimes})_{\mathfrak{a}}}(A(\mathfrak{a}), A(\mathfrak{a})) \in \mathcal{C}$ exists. Then there exists a centralizer $\mathfrak{z}(\mathrm{id}_A) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ with underlying object*

$$\mathfrak{z}(\mathrm{id}_A)(\mathfrak{a}) \simeq \mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C}^{\otimes} \times_{Fin_*} \mathbb{E}_1^{\otimes})_{\mathfrak{a}}}(A(\mathfrak{a}), A(\mathfrak{a})),$$

and the action of the centralizer has underlying map given by the evaluation α of the morphism object on $A(\mathfrak{a})$. Further, the multiplication of the $\mathbb{E}_1$-algebra structure on $\mathfrak{z}(\mathrm{id}_A)$ is induced by the action of the tensor product $\mathfrak{z}(\mathrm{id}_A)(\mathfrak{a}) \otimes \mathfrak{z}(\mathrm{id}_A)(\mathfrak{a})$ on $A(\mathfrak{a})$ given by the tensor product $\alpha \otimes \alpha$ in $\mathcal{E}_{\mathfrak{a}}$:

$$A(\mathfrak{a}) \xrightarrow{\simeq} A(\mathfrak{a}) \otimes A(\mathfrak{a}) \to (\mathfrak{z}(A)(\mathfrak{a}) \otimes A(\mathfrak{a})) \otimes (\mathfrak{z}(A)(\mathfrak{a}) \otimes A(\mathfrak{a})) \to A(\mathfrak{a}) \otimes A(\mathfrak{a}) \xrightarrow{\text{mult.}} A(\mathfrak{a}).$$

The proof of theorem 4.30 actually tells us much more than just the existence of the centralizer. Both $\mathbb{E}_1$-algebra structures on $\mathfrak{z}(A)$ are induced by actions of $\mathfrak{z}(A) \otimes \mathfrak{z}(A)$ on A by the property of the morphism object being final. In the above corollary, we obtained this action by knowing the monoidal structure in the double slice category.

To understand the algebra structure on the center, note that [Lur17, Theorem 2.2.2.4] gives a precise characterization of coCartesian lifts and hence tensor products in a parametrized undercategory. In particular, if $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads, and $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, then the induced map $q' : \mathcal{C}^{\otimes}_{A_{\mathcal{O}/}} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads. If further A is a trivial $\mathcal{O}^{\otimes}$-algebra, q' is even a coCartesian fibration of ∞-operads, and a morphisms F over f in $\mathcal{O}^{\otimes}$

$$\begin{tikzcd}
\Delta^1 \times \Delta^{\{0\}} \arrow[r, "\iota"] \arrow[d, "f"'] & \Delta^1 \times \Delta^1 \arrow[r, "p_1"] \arrow[d, "F"] & \Delta^1 \arrow[d, "f"] \\
\mathcal{O}^{\otimes} \times \Delta^{\{0\}} \arrow[r, "A"'] & \mathcal{C}^{\otimes} \arrow[r, "q"'] & \mathcal{O}^{\otimes}
\end{tikzcd}$$

is q'-coCartesian if and only if $F|_{\Delta^1 \times \Delta^{\{1\}}}$ is q-coCartesian. In particular, if $\mathcal{C}^{\otimes}$ is monoidal, the underlying object tensor product in the parametrized slice category is the tensor product in $\mathcal{C}^{\otimes}$.

Theorem 4.32. *Let $\mathcal{O}^{\otimes} = \mathbb{E}_1^{\otimes}$ and let $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ as above. Warning: Missing parts.*

4.5 The category of $\mathbb{E}_1$-modules in chain complexes

As explained above, we want to apply this corollary to the symmetric monoidal ∞-category $C^{\otimes} = N_{\mathrm{dg}}(\mathrm{Ch}(k))$ and $\tilde{A} \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. Note that $\mathbb{E}_2^{\otimes}$ only has a single color $\mathfrak{a}$, and by the above theorem the underlying object of the center $\mathfrak{z}(\tilde{A})$ at this color is equivalent to the morphism object $\mathrm{Mor}_{\overline{\mathcal{C}}_{\mathfrak{a},\mathfrak{m}}}(\tilde{A}, \tilde{A}) \in \overline{\mathcal{C}}_{\mathfrak{a},\mathfrak{a}}$. Here $\tilde{A}$ is viewed as a module over itself, i.e. as an object of

$$\overline{\mathcal{C}}_{\mathfrak{a},\mathfrak{m}} = \mathrm{Mod}_{\tilde{A}}^{\mathbb{E}_1}(\mathcal{C} \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}}.$$

Note that $\overline{\mathcal{C}}_{\mathfrak{a},\mathfrak{a}} = \mathrm{Mod}_1^{\mathbb{E}_1}(\mathcal{C} \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}} \simeq (\mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1^{\otimes})_{\mathfrak{a}} \simeq \mathcal{C}$, so

$$\mathfrak{z}(\tilde{A})(\mathfrak{a}) \simeq \mathrm{Mor}_{\mathrm{Mod}_{\tilde{A}}^{\mathbb{E}_1}(\mathcal{C} \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}}}(\tilde{A}, \tilde{A}) \in \mathcal{C}.$$

We are hence reduced to showing that $\mathcal{H}om_{\mathrm{Ch}(A^e)}(P,P)$ together with the evaluation map is such a morphism object, meaning that it satisfies the universal property of 2.3.

To show this, we must understand the ∞-category $\mathrm{Mod}^{\mathbb{E}_1}_{\tilde{A}}(N_{\mathrm{dg}}(\mathrm{Ch}(k)) \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}}$. By [Hin13, B.1.2], we have an equivalence of ∞-categories

$$\mathrm{Mod}^{\mathbb{E}_1}_{\tilde{A}}(N_{\mathrm{dg}}(\mathrm{Ch}(k)) \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}} \simeq \mathrm{Alg}_{M\mathbb{E}_1}(N_{\mathrm{dg}}(\mathrm{Ch}(k))) \times_{\mathrm{Alg}_{\mathbb{E}_1}(N_{\mathrm{dg}}(\mathrm{Ch}(k)))} \{\tilde{A}\}.$$

We can now use Hinich's rectification theorem for modules [Hin13, 5.2.3] to get an equivalence of ∞-categories

$$N(\mathrm{Mod}^{C_*(\mathbb{E}_1)}_{\tilde{A}}(\mathrm{Ch}(k))^c)[W^{-1}_{\mathrm{Mod}}] \xrightarrow{\simeq} \mathrm{Mod}^{\mathbb{E}_1}_{\tilde{A}}(N_{\mathrm{dg}}(\mathrm{Ch}(k)) \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}}.$$

Note here that since $\tilde{A}$ is cofibrant, by [BM08, 2.6] the module category $\mathrm{Mod}^{C_*(\mathbb{E}_1)}_{\tilde{A}}(\mathrm{Ch}(k))$ indeed carries a model category structure transferred via the forgetful functor to $\mathrm{Ch}(k)$. Similarly, by [BM08, 2.7], the category $\mathrm{LMod}_{U_{C_*(\mathbb{E}_1)}(\tilde{A})}(\mathrm{Ch}(k))$ can be made into a model category via transfer from the forgetful functor. By [BM08, 1.10] we have an isomorphism of categories making the following diagram commute

$$\begin{tikzcd}
\mathrm{Mod}^{C_*(\mathbb{E}_1)}_{\tilde{A}}(\mathrm{Ch}(k)) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{LMod}_{U_{C_*(\mathbb{E}_1)}(\tilde{A})}(\mathrm{Ch}(k)) \arrow[dl] \\
& \mathrm{Ch}(k) &
\end{tikzcd}$$

In particular, this isomorphism yields a Quillen equivalence between these two categories.

4.6 The trouble with universal enveloping algebras

We now want to relate this category of left modules over $U_{C_*(\mathbb{E}_1)}(\tilde{A})$ to the category $\mathrm{Ch}(A^e)$. To this end, we have the following result.

Proposition 4.33. *There exists a zig-zag of quasi-isomorphisms between $U_{C_*(\mathbb{E}_1)}(\tilde{A})$ and $A \otimes_k A^{\mathrm{op}}$.*

Proof. Let $\theta : \mathbb{A}_\infty \xrightarrow{\simeq} \mathrm{Assoc}$ be a cofibrant replacement, and note that we have a diagram

$$\begin{tikzcd}
& C_*(\mathbb{E}_1) \arrow[d, "\phi,\simeq"] \\
\mathbb{A}_\infty \arrow[ur, "\psi,\simeq"] \arrow[r, two heads, "\theta,\simeq"'] & \mathrm{Assoc}
\end{tikzcd}$$

Since we work in characteristic zero, all of the involved operads are admissible and Σ-cofibrant. In particular, all the above weak equivalences are strong equivalences of operads, and thus induce a Quillen equivalence between their respective algebra categories. Let $A' \xrightarrow{\simeq} A$ be a cofibrant replacement in associative algebras, and let $\hat{A} \xrightarrow{\simeq} \theta^* A'$ be a cofibrant replacement in $\mathbb{A}_\infty$-algebras. Then in particular, the unit map $\hat{A} \to \psi^*\psi_!\hat{A}$ is a weak equivalence, and hence using [Fre09, Theorem 17.4.A, 17.4.B] we get the following diagram of weak equivalences of dg algebras

$$\begin{tikzcd}[column sep=small]
 & & & U_{\mathbb{A}_\infty}(\hat{A}) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & \\
 & & U_{\mathbb{A}_\infty}(\theta^* A') \arrow[dl, "\theta_\flat"'] & & U_{\mathbb{A}_\infty}(\psi^*\psi_! \hat{A}) \arrow[dr, "\psi_\flat"] & \\
 & A' \otimes_k A'^{\mathrm{op}} = U_{\mathrm{Assoc}}(A') \arrow[dl, "\simeq"'] & & & & U_{C_*(\mathbb{E}_1)}(\psi_! \hat{A}) \\
A \otimes_k A^{\mathrm{op}} & & & & &
\end{tikzcd}$$

It hence suffices to show that $U_{C_*(\mathbb{E}_1)}(\psi_! \hat{A})$ is quasi-isomorphic to $U_{C_*(\mathbb{E}_1)}(\tilde{A})$. To this end, let $A_1 \xrightarrow{\simeq} \phi^* A'$ be any cofibrant replacement. Note that $\psi_! \hat{A}$ is again cofibrant, and we hence have a lift $f : \psi_! \hat{A} \to A_1$ in the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A_1 \arrow[d, two heads, "\simeq"] \\
\psi_! \tilde{A} \arrow[r, "\simeq"'] \arrow[ur, dashed, "f"] & \phi^* A'
\end{tikzcd}$$

and by 2-out-of-3, f must be a weak equivalence. This is a weak equivalence between cofibrant objects, so again by [Fre09, Theorem 17.4.A], we get a quasi-isomorphism $U_{C_*(\mathbb{E}_1)}(\psi_! \hat{A}) \xrightarrow{\simeq} U_{C_*(\mathbb{E}_1)}(A_1)$. The map $\phi^* A' \to \phi^* A$ is again a trivial fibration as ϕ^* is right Quillen, and in particular the composition

$$A_1 \overset{\simeq}{\twoheadrightarrow} \phi^* A' \overset{\simeq}{\twoheadrightarrow} \phi^* A$$

is again a cofibrant replacement. We now again find a lift $g : A_1 \to \tilde{A}$ in the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \tilde{A} \arrow[d, two heads, "\simeq"] \\
A_1 \arrow[r, two heads, "\simeq"'] \arrow[ur, dashed, "g"] & \phi^* A
\end{tikzcd}$$

which again is a weak equivalence, and thus finally induces a quasi-isomorphism

$$U_{C_*(\mathbb{E}_1)}(A_1) \xrightarrow{\simeq} U_{C_*(\mathbb{E}_1)}(\tilde{A}).$$

Summarizing, we get the following zig-zag

$$A \otimes_k A^{\mathrm{op}} \xleftarrow{\simeq} U_{\mathbb{A}_\infty}(\hat{A}) \xrightarrow{\simeq} U_{C_*(\mathbb{E}_1)}(\tilde{A}).$$

$\square$

Corollary 4.34. *Let $\tilde{A} \overset{\simeq}{\twoheadrightarrow} \phi^* A$ be a cofibrant replacement of the associative algebra A as an $C_*(\mathbb{E}_1)$-algebra. Then the category $\mathrm{LMod}_{A \otimes_k A^{\mathrm{op}}}(\mathrm{Ch}(k))$ is Quillen equivalent to the category $\mathrm{LMod}_{U_{C_*(\mathbb{E}_1)}(\tilde{A})}(\mathrm{Ch}(k))$.*

Finally, we have an isomorphism

$$\begin{tikzcd}
\mathrm{LMod}_{A\otimes_k A^{\mathrm{op}}}(\mathrm{Ch}(k)) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Ch}(A\otimes_k A^{\mathrm{op}}) \arrow[dl] \\
& \mathrm{Ch}(k) &
\end{tikzcd}$$

which again induces a Quillen equivalence. In particular, the two model categories $\mathrm{Ch}(A \otimes_k A^{\mathrm{op}})$ and $\mathrm{Mod}_{\tilde{A}}^{C_*(\mathbb{E}_1)}(\mathrm{Ch}(k))$ are Quillen equivalence and therefore present the same ∞-category.

Corollary 4.35. *There is an equivalence of ∞-categories*

$$N_{\mathrm{dg}}(\mathrm{Ch}(A\otimes_k A^{\mathrm{op}})^\circ) \xrightarrow{\simeq} \mathrm{Mod}_{\tilde{A}}^{\mathbb{E}_1}(N_{\mathrm{dg}}(\mathrm{Ch}(k)) \times_{N(Fin_*)} \mathbb{E}_1)_{\mathfrak{a}}.$$

Proof. This follows from the above discussion together with the fact that $\mathrm{Ch}(A \otimes A^{\mathrm{op}})$ satisfies the conditions of corollary 4.21. ☐

4.7 Morphism objects in dg model categories

To proof theorem 4.27, we want to argue that $\mathcal{H}om_{\mathrm{Ch}\, A^e}(P, P)$ is an endomorphism object of $\tilde{A}$ in the $\mathbb{E}_1$-module category over $\tilde{A}$. We have already seen that this module category is equivalent to the ∞-category $N_{\mathrm{dg}}(\mathrm{Ch}(A^e)^\circ)$, and we know that $\mathcal{H}om_{\mathrm{Ch}\, A^e}(P, P) \in \mathrm{Ch}(k)$ is indeed an endomorphism object for $P \simeq A$ in the dg category $\mathrm{Ch}(A^e)$. To finish the argument, we will now proof the following general theorem connecting morphism objects in a dg category and its underlying ∞-category.

Lemma 4.36. *If $\mathcal{C}$ is a monoidal dg model category with underlying monoidal product $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$. Then the induced monoidal product $N_{dg}(\mathcal{C}^\circ) \times N_{dg}(\mathcal{C}^\circ) \to N_{dg}(\mathcal{C}^\circ)$ sends $A, B \in \mathcal{C}^\circ$ to an object equivalent to $R(A \otimes B)$. A similar statement holds for dg model categories left tensored over a monoidal dg model category.*

Proof. Missing parts! ☐

Theorem 4.37. *Let $\mathcal{C}$ be a monoidal dg model category and let $\mathcal{M}$ be a dg model category that is left tensored over $\mathcal{C}$. In particular, we have a dg functor $\otimes : \mathcal{C} \boxtimes \mathcal{M} \to \mathcal{M}$ whose underlying functor is a left Quillen bifunctor. Assume that for $A, B \in \mathcal{M}^\circ$ we have a dg morphism object $\mathcal{H}om_{\mathcal{M}}(A, B) \in \mathcal{C}$ together with map $\alpha : \mathcal{H}om_{\mathcal{M}}(A, B) \otimes A \to B$ in $\mathcal{M}$ such that composition with α induces an isomorphism*

$$\mathrm{Map}_{\mathcal{C}}(C, \mathcal{H}om_{\mathcal{M}}(A, B)) \cong \mathrm{Map}_{\mathcal{M}}(C \otimes A, B). \tag{2}$$

Then

1. *(1) The induced map $\tilde{\alpha} \in \mathrm{Map}_{N_{\mathrm{dg}}(\mathcal{M}^\circ)}(R(Q\mathcal{H}om_{\mathcal{M}}(A, B) \otimes A), B)$ makes $Q\mathcal{H}om_{\mathcal{M}}(A, B) \in N_{\mathrm{dg}}(\mathcal{C}^\circ)$ into a morphism object for $A, B \in N_{\mathrm{dg}}(\mathcal{M}^\circ)$ in the sense of definition 2.3.*

2. *(2) If $\beta : R(M \otimes A) \to B$ is another morphism object for A and B in $N_{\mathrm{dg}}(\mathcal{M}^\circ)$, then $f : M \xrightarrow{\simeq} Q\mathcal{H}om_{\mathcal{M}}(A, B)$ is a weak equivalence in $\mathcal{C}$, and $\tilde{\alpha} \circ R(f \otimes \mathrm{id}_A) \simeq \beta$ are chain homotopic.*

Proof. For (1), note that if $C \in N_{\mathrm{dg}}(\mathcal{C}^\circ)$ is bifibrant, $Q\mathcal{H}om_{\mathcal{M}}(A, B) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{M}}(A, B)$ is the cofibrant replacement map, and $C \otimes A \xhookrightarrow{\simeq} R(C \otimes A)$ is the fibrant replacement map, we get a weak equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, Q\mathcal{H}om_{\mathcal{M}}(A, B)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(C, \mathcal{H}om_{\mathcal{M}}(A, B)) \cong \mathrm{Map}_{\mathcal{M}}(C \otimes A, B) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{M}}(R(C \otimes A), B)$$

of chain complexes. Applying $\mathrm{DK}_\bullet \tau_{\geq 0}$, we get

$$\mathrm{Map}_{N_{\mathrm{dg}}(\mathcal{C}^\circ)}(C, Q\mathcal{H}om_{\mathcal{M}}(A,B)) \simeq \mathrm{Map}_{N_{\mathrm{dg}}(\mathcal{M}^\circ)}(R(C \otimes A), B)$$

together with lemma 4.36 this yields the result.
For (2), we automatically get $M \simeq \mathcal{H}om_{\mathcal{M}}(A,B)$ in the ∞-category $N_{\mathrm{dg}}(\mathcal{C}^\circ)$ since morphism objects are unique up to equivalence. Now recall that $N_{\mathrm{dg}}(\mathcal{C}^\circ) \simeq N(\mathcal{C}^c)[W^{-1}]$, and since model categories are saturated this implies the result. □

This finally allows us to proof the main theorem of this section.

Proof of theorem 4.27. By theorem **??**, it suffices to show that $\mathrm{ev} : \mathcal{H}om_{\mathrm{Ch}(A^e)}(P,P) \otimes_k P \to P$ makes $\mathcal{H}om_{\mathrm{Ch}(A^e)}(P,P)$ into an endomorphism object for $\tilde{A}$ in the ∞-category $\mathrm{Mod}_{\tilde{A}}^{\mathbb{E}_1}(\mathcal{C} \times_{N(\mathrm{Fin}_*)} \mathbb{E}_1)_{\mathfrak{a}}$. By corollary 4.35, this ∞-category is equivalent to the ∞-category $N_{\mathrm{dg}}(\mathrm{Ch}(A^e)^\circ)$. Note that $\mathcal{M} = \mathrm{Ch}(A^e)$ satisfies the conditions of theorem 4.37 with $\mathcal{C} = \mathrm{Ch}(k)$, and $P \in \mathcal{M}$ is indeed cofibrant (and thus bifibrant). Therefore, the evaluation map above indeed makes $\mathcal{H}om_{\mathrm{Ch}(A^e)}(P,P)$ into a morphism object for $P \in N_{\mathrm{dg}}(\mathrm{Ch}(A^e)^\circ)$. Clearly, P is equivalent to $\tilde{A}$ viewed in the $\mathbb{E}_1$-module category, and we hence get our result. □

5 The Hochschild complex of a scheme

We would now like to globalize the above results and consider a quasi-compact separable scheme X over a field k with $\mathbb{Q} \subseteq k$. In particular, we suggest that the $\mathbb{E}_1$-center of the structure sheaf is the morally correct way to define the Hochschild complex of such a scheme. Note that this does not require X to be smooth.

5.1 The ∞-category of sheaves of k-modules

Let X be a scheme as above. Then $\mathcal{O}_X$ is an associative algebra in the category of sheaves of k-modules on X. We again want to view $\mathcal{O}_X$ as a $\mathbb{E}_1$-algebra in the associated ∞-category.

Proposition 5.38 ([Hin05], Theorem 1.3.1)**.** *Let S be a site. There is a cofibrantly generated model structure on the category $\mathrm{Ch}(\hat{S}_k)$ of presheaves of k-module complexes on S with*

- *weak equivalences the maps $f : \mathcal{F} \to \mathcal{G}$ such that the sheafification $f^a : \mathcal{F}^a \to \mathcal{G}^a$ is a quasi-isomorphism of sheaves,*
- *cofibrations generated by maps $f : \mathcal{F} \to \mathcal{F}\langle x; dx = z \in \mathcal{F}(U)\rangle$ corresponding to adding a section to kill a cycle z over $U \in S$, and*
- *fibrations the maps $f : \mathcal{F} \to \mathcal{G}$ such that $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all $U \in S$ and for any hypercover $\epsilon : V_\bullet \to U$ the diagram*

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "f(U)"'] & \check{C}(V_\bullet, \mathcal{F}) \arrow[d] \\
\mathcal{G}(U) \arrow[r] & \check{C}(V_\bullet, \mathcal{G})
\end{tikzcd}$$

is a homotopy pullback.

This is the left Bousfield localization of the projective model structure on $\mathrm{Ch}(\hat{S}_k)$ with respect to the Čech complexes of hypercoverings. We therefore call it the local projective model structure. Note that if S has enough points, weak equivalences can be detected at stalks.

Definition 5.39. Let X be a scheme over k. We consider the following two sites naturally associated to X. Let $\mathrm{Aff}(X)$ be the site of affine open subsets of X, and let $\mathrm{Open}(X)$ the site of all open subsets of X. We have a natural inclusion $\iota : \mathrm{Aff}(X) \to \mathrm{Open}(X)$ that induces to a restriction functor

$$\iota_* : \mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k) \to \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)$$

Proposition 5.40. *The restriction functor ι_* admits a left adjoint ι^{-1} and the pair $\iota^{-1} \dashv \iota_*$ forms a Quillen equivalence. Both ι_* and ι^{-1} preserve weak equivalences, and ι^{-1} preserves acyclic fibrations. The unit $\mathrm{id} \Rightarrow \iota_*\iota^{-1}$ is an isomorphism, and the counit $\iota^{-1}\iota_* \Rightarrow \mathrm{id}$ is a component-wise weak equivalence.*

Proof. The left adjoint ι^{-1} is given by $\iota^{-1}\mathcal{F}(V) = \mathrm{colim}_{V \subseteq U \in \widehat{\mathrm{Aff}(X)}} \mathcal{F}(U)$. The direct image ι_* clearly preserves acyclic fibrations, since these are pointwise. The sites of all opens and of affine opens have the same points, namely points in the topological space X. This follows because affine opens form a basis of the Zariski topology. Even more, ι_* and ι^{-1} preserve stalks at these points. Taking stalks is a left adjoint, so this follows trivially for the inverse image, and for the direct image we note that small enough neighborhoods of a point $x \in X$ always contain an affine open neighborhood of x. This shows that both adjoints preserve weak equivalences, and in particular ι^{-1} preserves acyclic cofibrations. The fact that ι^{-1} preserves acyclic fibrations follows from the fact that filtered colimits are exact in Grothendieck categories. Finally, note that if U is affine, then $\mathrm{colim}_{U \subseteq W \in \widehat{\mathrm{Aff}(X)}} \mathcal{F}(W) \cong F(U)$ since U is final in the index category. This shows that the unit is an isomorphism. The fact that the counit is a component-wise weak equivalence again follows from the fact that both adjoints preserve weak equivalences. □

Definition 5.41. We call the underlying ∞-category of the local projective model structure on $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$ is the ∞-category of sheaves of k-modules on X

$$\mathrm{Sh}_\infty(X) := N(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)^c)[W^{-1}].$$

By proposition 5.40, we have an equivalence of ∞-categories

$$\iota_* : N(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)^c)[W^{-1}] \to \mathrm{Sh}_\infty(X)$$

with quasi-inverse $\iota^{-1} : \mathrm{Sh}_\infty(X) \to N(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)^c)[W^{-1}]$.

Remark 5.42. Even though the model categories of presheaves on affine opens and general opens yield the same ∞-category, the above model category structure depends on the choice of site. On affine open subsets, all quasi-coherent sheaves on X automatically fibrant, which is not true for general opens. In particular, on affine opens the structure sheaf $\mathcal{O}_X$ itself is fibrant.

Proposition 5.43. *If the tops on S has enough points and S admits finite products, then the local projective model structure yields a closed symmetric monoidal model category. If in addition S admits a final object, then $\mathrm{Ch}(\hat{S}_k)$ is a dg symmetric monoidal model category.*

Proof. By [PS14, Prop. 7.9], the global projective model structure on $\mathrm{Ch}(\hat{S}_k)$ inherits the structure of a (symmetric) monoidal model category since S admits finite products. By [Whi14, Thm. 4.6], to show that this monoidal model structure descends to the local projective model structure, it suffices to argue that for f a local weak equivalence and $\mathcal{F}$ a cofibrant object, the map $f \otimes \mathrm{id}_{\mathcal{F}}$ is again a local weak equivalence. But this is clear if the topos has enough points, since we can then check local weak equivalences on stalks. Since the presheaf category admits an internal hom, this shows the first part. For the second part, note that presheaves of chain complexes are the same as chain complexes of presheaves of k-modules. Since the later is an abelian category, this automatically admits a dg enrichment. Recall that if $* \in S$ is terminal, we have the constant presheaf functor

$$\begin{aligned} C_* : \mathrm{Ch}(k) &\to \mathrm{Ch}(\hat{S}_k) \\ C &\mapsto (U \mapsto C) \end{aligned}$$

By the argument below, this functor preserves cofibrations. We can hence define a tensoring

$$\mathrm{Ch}(k) \times \mathrm{Ch}(\hat{S}_k) \to \mathrm{Ch}(\hat{S}_k), \quad (C, \mathcal{F}) \mapsto C_*(C) \otimes \mathcal{F}$$

as well as a powering

$$\mathrm{Ch}(k)^{\mathrm{op}} \times \mathrm{Ch}(\hat{S}_k) \to \mathrm{Ch}(\hat{S}_k), \quad (C, \mathcal{F}) \mapsto \mathcal{H}om(C_*(C), \mathcal{F}).$$

One easily checks that these indeed satisfy the correct adjointness properties. It hence suffices to check the pushout-product axiom. But if $i : C \to D$ is a cofibration in $\mathrm{Ch}(k)$, then $C_*(i) : C_*(C) \to C_*(D)$ is a cofibration in $\mathrm{Ch}(\hat{S}_k)$, and therefore this follows directly from the pushout-product axiom in $\mathrm{Ch}(\hat{S}_k)$. $\square$

Let $U \subseteq X$ be an affine open. Then we have adjoint functors

$$\begin{tikzcd} \mathrm{Ch}(k) \arrow[r, shift left, "C_U"] \arrow[r, phantom, "\perp"] & \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k). \arrow[l, shift left, "\Gamma_U"] \end{tikzcd}$$

where Γ_U sends a complex of presheaves $\mathcal{F}$ to $\mathcal{F}(U)$ and C_U is the constant presheaf functor sending C to the presehaf

$$V \mapsto \begin{cases} C & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}.$$

If we equip $\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)$ with the projective model structure, then Γ_U preserves fibrations and weak equivalences by construction. Therefore we obtain a Quillen adjunction. We can compose this with the Quillen adjucntion

$$\begin{tikzcd} \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)^{\mathrm{proj}} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, phantom, "\perp"] & \mathrm{Ch}\,\widehat{\mathrm{Aff}(X)}_k)^{\mathrm{loc}} \arrow[l, shift left, "\mathrm{id}"] \end{tikzcd}$$

of the Bousfield localization to obtain a Quillen adjunction

$$\begin{tikzcd} \mathrm{Ch}(k) \arrow[r, shift left, "C_U"] \arrow[r, phantom, "\perp"] & \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)^{\mathrm{loc}}. \arrow[l, shift left, "\Gamma_U"] \end{tikzcd}$$

Since C_U is left Quillen, it preserves weak equivalences between cofibrant objects. But every object in $\mathrm{Ch}(k)$ is cofibrant, so C_U preserves weak equivalences.

The same argument works if we instead consider the site of all opens $\mathrm{Open}(X)$. In this case, for any open $V \subseteq X$ we obtain a Quillen adjunction $C_V \dashv \Gamma_V$. Note that in particular we can then take $U = X$. If U is affine, this agrees with the above construction.

The functors C_V and Γ_V are both strong monoidal, since the tensor product of presheaves is takes section-wise. In particular, we obtain lax monoidal functors of ∞-categories $C_V : \mathcal{D}_\infty(k)^\otimes \to \mathrm{Sh}_\infty(X)^\otimes$ and $R\Gamma_V : \mathrm{Sh}_\infty(X)^\otimes \to \mathcal{D}_\infty(k)^\otimes$.

Definition 5.44. Let $\mathcal{O}$ be a dg operad. The corresponding operad in $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$ is given by $C_X(\mathcal{O})$. By abuse of notation, we will usually denote the operad $C_X(\mathcal{O})$ just by $\mathcal{O}$.

Lemma 5.45. *The functor C_X preserves cofibrancy and Σ-cofibrancy of operads, as well as weak equivalences of operads. Every operad in $\mathrm{Ch}(\hat{X}_k)$ is admissible, and even strongly admissible if it is in the image of C_X.*

Proof. The Quillen adjunction $C_X \dashv \Gamma_X$ induces adjunctions between the respective categories of symmetric collections and symmetric operads, since both are strong symmetric monoidal. The model structure on symmetric collection is transferred from the underlying model category, and hence the adjunction is again Quillen. Similarly, fibrations and weak equivalences of operads are pointwise, and hence Γ_X preserves fibrations and trivial fibrations of operads. To see that C_X preserves weak equivalences, note that these are point-wise in operads, and C_X preserves weak equivalences on the underlying model categories. To see that operads in $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$ are admissible, use [PS14, Theorem 5.11] and section 8 of [PS18]. Now to see that every operad in the image of C_X is even strongly admissible, use [PS14, Proposition 6.3] together with the fact that any operad in $\mathrm{Ch}(k)$ is Σ-cofibrant. ☐

This shows that we get a diagram of admissible Σ-cofibrant operads

$$\begin{tikzcd} & C_X(C_*(\mathbb{E}_1)) \arrow[d, "{C_X(\phi),\simeq}"] \\ C_X(\mathbb{A}_\infty) \arrow[ur, "{C_X(\psi),\simeq}"] \arrow[r, "{C_X(\theta),\simeq}"'] & C_X(\mathrm{Assoc}) \end{tikzcd}$$

and $C_X(\mathbb{A}_\infty)$ is still cofibrant.

We can hence form the $C_*(\mathbb{E}_1)$-algebra $\phi^*\mathcal{O}_X$ and choose a cofibrant replacement $\tilde{\mathcal{O}}_X \xrightarrow{\simeq} \phi^*\mathcal{O}_X$. Then we have $\tilde{\mathcal{O}}_X \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sh}_\infty(X))$ and can consider the center

$$\mathfrak{z}(\tilde{\mathcal{O}}_X) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sh}_\infty(X))) \simeq \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sh}_\infty(X)).$$

The rectification theorem 4.25 applies to $\mathbb{E}_2$ algebras in the ∞-category of sheaves on X. In particular, $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$ is symmetrically flat by the proof of lemma 5.45. This means that we can strictify this $\mathbb{E}_2$-algebra structure to the category of complexes of presheaves.

The remainder of this chapter is dedicated to arguing that this center is the correct Hochschild complex of X. In particular, we will show that for a smooth scheme this recovers the sheaf of polydifferential operators.

5.2 The Hochschild complex of a scheme is local

In this section we will prove the following theorem, showing that the center of a scheme satisfies a homotopy descent condition.

Theorem 5.46. *Let $U = \mathrm{Spec}(A) \subseteq X$ be an affine open. The map $R\Gamma_U : \mathrm{Sh}_\infty(X) \to \mathcal{D}_\infty(k)$ is lax symmetric monoidal and hence induces a map $R\Gamma_U : \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sh}_\infty(X)) \to \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}_\infty(k))$. We have*

$$R\Gamma_U(\mathfrak{z}(\tilde{\mathcal{O}}_X)) \simeq \mathfrak{z}(\tilde{A}) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}_\infty(k))$$

for any cofibrant replacement $\tilde{A}$ of A.

Just like in the affine case, we have

$$\mathfrak{z}(\tilde{\mathcal{O}}_X)(\mathfrak{a}) \simeq \mathrm{Mor}_{\mathrm{Mod}^{\mathbb{E}_1}_{\tilde{\mathcal{O}}_X}(\mathrm{Sh}_\infty(X)\times_{Fin_*}\mathbb{E}_1)_\mathfrak{a}}(\tilde{\mathcal{O}}_X, \tilde{\mathcal{O}}_X) \in \mathrm{Sh}_\infty(X),$$

and it hence suffices to understand this endomorphism object. To this end, note that we can adapt Hinich's rectification theorem for modules [Hin13, 5.2.3] to the local projective model structure on complexes of presheaves to get an equivalence of ∞-categories

$$N(\mathrm{Mod}^{C_*(\mathbb{E}_1)}_{\tilde{\mathcal{O}}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))^c)[W^{-1}_{\mathrm{Mod}}] \simeq \mathrm{Mod}^{\mathbb{E}_1}_{\tilde{\mathcal{O}}_X}(\mathrm{Sh}_\infty(X) \times_{Fin_*} \mathbb{E}_1)_\mathfrak{a}.$$

Again following the affine case, we have a Quillen equivalence

$$\mathrm{Mod}^{C_*(\mathbb{E}_1)}_{\tilde{\mathcal{O}}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)) \cong \mathrm{LMod}_{U_{C_*(\mathbb{E}_1)}(\tilde{\mathcal{O}}_X)}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)).$$

Proposition 5.47. *There exists a zig-zag of weak equivalences betwen $U_{C_*(\mathbb{E}_1)}(\tilde{\mathcal{O}}_X)$ and $\mathcal{O}_X \otimes \mathcal{O}_X$ in the category of associative algebras in $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$.*

Proof. We adapt the proof of proposition 4.33. Let $\mathcal{O}'_X \xrightarrow{\simeq} \mathcal{O}_X$ be a cofibrant resolution of Assoc-algebras in $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$. Then

$$(\mathcal{O}'_X \otimes \mathcal{O}'_X)_x \xrightarrow{\cong} \mathcal{O}'_{X,x} \otimes \mathcal{O}'_{X,x} \xrightarrow{\simeq} \mathcal{O}_{X,x} \otimes \mathcal{O}_{X,x} \xrightarrow{\cong} (\mathcal{O}_X \otimes \mathcal{O}_X)_x,$$

showing that $\mathcal{O}'_X \otimes \mathcal{O}'_X$ is weakly equivalent to $\mathcal{O}_X \otimes \mathcal{O}_X$. The rest of the argument goes through exactly like before with the following amendment: Let $\hat{\mathcal{O}}_X \xrightarrow{\simeq} \theta^*\mathcal{O}'_X$ be a cofibrant replacement of $\mathbb{A}_\infty$-algebras. To obtain the zig-zag

$$\begin{tikzcd} & U_{\mathbb{A}_\infty}(\hat{\mathcal{O}}_X) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ U_{\mathbb{A}_\infty}(\theta^*\mathcal{O}'_X) & & U_{\mathbb{A}_\infty}(\psi^*\psi_!\hat{\mathcal{O}}_X) \end{tikzcd}$$

we have to argue that the underlying complexes of presheaves of these $\mathbb{A}_\infty$-algebras are cofibrant. To this end, recall from lemma 5.45 that $\mathbb{A}_\infty$ and Assoc are both strongly admissible, meaning that the forgetful functor from algebras preserves cofibrant objects. In particular, the underlying complexes of presheaves of $\mathcal{O}'_X$ and $\hat{\mathcal{O}}_X$ are both cofibrant. Now recall that $\psi_!$ is left Quillen and hence preserves cofibrancy, and the restriction of scalars functors θ^* and ψ^* do not alter the underlying complex. This finishes the argument. □

In particular, the category $\mathrm{LMod}_{U_{C_*(\mathbb{E}_1)}(\tilde{\mathcal{O}}_X)}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))$ is Quillen equivalent to the category $\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))$.

Lemma 5.48. *The category* $\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))$ *is a dg model category that is left tensored over* $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$*). For* $\mathcal{M},\mathcal{N}\in\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))$*, we have a morphism object*

$$\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{M},\mathcal{N})(V)=\mathrm{Hom}_{\mathcal{O}_V\otimes\mathcal{O}_V}(\mathcal{M}|_V,\mathcal{N}|_V)\in\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k).$$

Proof. Since $\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)})_k))$ is the category of complexes of presheaves of $\mathcal{O}_X\otimes\mathcal{O}_X$-modules, the dg enrichment is clear. Now note that $\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)})_k))$ is tensored and powered over $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$: If $\mathcal{F}\in\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$ and $\mathcal{M}\in\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)})_k))$, we obtain an $\mathcal{O}_X\otimes\mathcal{O}_X$-module structure on the tensor product in complexes of presheaves by

$$(\mathcal{O}_X\otimes\mathcal{O}_X)\otimes(\mathcal{F}\otimes\mathcal{M})\cong\mathcal{F}\otimes\mathcal{O}_X\otimes\mathcal{O}_X\otimes\mathcal{M}\to\mathcal{F}\otimes\mathcal{M}.$$

We obtain a module structure on $\mathrm{Hom}_{\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)}(\mathcal{F},\mathcal{M})$ by the pointwise module structure

$$\begin{aligned}\mathcal{O}_X\otimes\mathcal{O}_X\otimes\mathrm{Hom}_{\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)}(\mathcal{F},\mathcal{M})\to\mathrm{Hom}_{\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)}(\mathcal{M},\mathcal{M})\otimes\mathrm{Hom}_{\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)}(\mathcal{F},\mathcal{M})\\ \xrightarrow{\text{composition}}\mathrm{Hom}_{\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)}(\mathcal{F},\mathcal{M}).\end{aligned}$$

One easily checks that these indeed yield a tensoring and powering respectively. Now to make the module category into a dg category, we precompose these operations with the constant presheaf functor $C_X:\mathrm{Ch}(k)\to\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$. The pushout-product axiom is checked in [Hin05, Lemma 1.6.3], and Hinich also shows in the same section that the above yields a morphism object. ☐

We can now again use theorem 4.37 to conclude that the center of $\tilde{\mathcal{O}}_X$ in the ∞-category $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sh}_\infty(X))$ is given by

$$\mathfrak{z}(\tilde{\mathcal{O}}_X)(\mathfrak{a})\simeq Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O},\mathcal{O})$$

for a bifibrant model $\mathcal{O}$ of $\mathcal{O}_X$ as an $\mathcal{O}_X\otimes\mathcal{O}_X$-module. The center action is given by the evaluation map

$$R(Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O},\mathcal{O})\otimes\mathcal{O})\to\mathcal{O}.$$

This is in contrast to the affine case, where A was already fibrant as an $A\otimes A$-module. The reason for this is that $\mathcal{O}_X$ is not fibrant in the local projective model structure on $\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)$. It is however fibrant in the local projective model structure for the site of affine opens on X, and we have already seen that presheaves on this smaller site present the same ∞-category.

Proposition 5.49. *We have an equivalence*

$$\iota_*(Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O},\mathcal{O}))\simeq Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\mathcal{P})$$

for any cofibrant resolution $\mathcal{P}\xrightarrow{\simeq}\iota_*\mathcal{O}_X$ *in* $\mathrm{LMod}_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k))$.

Lemma 5.50. *We have a Quillen equivalence*

$$\mathrm{LMod}_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)).$$

Both adjoints preserve weak equivalences.

Proof. The Quillen equivalence $\iota^{-1} \dashv \iota_*$ induces a Quillen equivalence

$$\mathrm{LMod}_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)) \underset{\iota_*}{\overset{\iota^{-1}}{\rightleftarrows}} \mathrm{LMod}_{\iota^{-1}\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)).$$

The counit yields a weak equivalence $\epsilon : \iota^{-1}\iota_*(\mathcal{O}_X \otimes \mathcal{O}_X) \xrightarrow{\simeq} \mathcal{O}_X \otimes \mathcal{O}_X$, and hence we get a Quillen equivalence

$$\mathrm{LMod}_{\iota^{-1}\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)) \underset{\epsilon_*}{\overset{\epsilon^*}{\rightleftarrows}} \mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k)).$$

composing these yields the Quillen equivalence in the statement. Clearly, ϵ_* preserves weak equivalences. But ϵ is an isomorphism on stalks, and hence ϵ^* also preserves weak equivalences. □

Lemma 5.51. *For* $\mathcal{F} \in \mathrm{LMod}_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k))$ *and* $\mathcal{G} \in \mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Open}(X)}_k))$, *we have*

$$\iota_*\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\iota^*\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{F}, \iota_*\mathcal{G}).$$

Proof. Let $U \in \mathrm{Aff}(X)$, and let $\iota' : \mathrm{Aff}(U) \to \mathrm{Open}(U)$ be the inclusion. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_U\otimes\mathcal{O}_U}(\iota^*\mathcal{F}|_U, \mathcal{G}|_U) &\cong \mathrm{Hom}_{\mathcal{O}_U\otimes\mathcal{O}_U}(\iota'^*(\mathcal{F}|_U), \mathcal{G}|_U) \\ &\cong \mathrm{Hom}_{\iota'_*(\mathcal{O}_U\otimes\mathcal{O}_U)}(\mathcal{F}|_U, \iota'_*(\mathcal{G}|_U)) \\ &\cong \mathrm{Hom}_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)|_U}(\mathcal{F}|_U, \iota_*\mathcal{G}|_U). \end{aligned}$$

□

Proof of proposition 5.49. By definition, we have a diagram

$$\begin{tikzcd} \mathcal{P}' \arrow[r, two heads, "\simeq"] \arrow[d, tail, "\simeq"'] & \mathcal{O}_X \\ \mathcal{O} & \end{tikzcd}$$

with $\mathcal{P}'$ cofibrant and $\mathcal{O}$ bifibrant. If $\mathcal{P} \xrightarrow{\simeq} \iota_*\mathcal{O}_X$ is a cofibrant resolution, we get a weak equivalence $\iota^*\mathcal{P} \xrightarrow{\simeq} \iota^*\iota_*\mathcal{O}_X$ and $\iota^*\mathcal{P}$ is still cofibrant. We can hence solve the following lifting problem

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, tail] & & \mathcal{P}' \arrow[d, two heads, "\simeq"] \\ \iota^*\mathcal{P} \arrow[r, "\simeq"'] \arrow[rru, dashed] & \iota^*\iota_*\mathcal{O}_X \arrow[r, "\simeq"'] & \mathcal{O}_X \end{tikzcd}$$

and by 2-out-of-3, the map $\iota^*\mathcal{P} \to \mathcal{P}'$ is again a weak equivalence. Let $\iota^*\mathcal{P} \overset{\simeq}{\rightarrowtail} \mathcal{R}$ be a fibrant resolution. We can also solve the lifting problem

$$\begin{tikzcd} \iota^*\mathcal{P} \arrow[r, "\simeq"] \arrow[d, tail, "\simeq"'] & \mathcal{P}' \arrow[r, tail, "\simeq"] & \mathcal{O} \arrow[d] \\ \mathcal{R} \arrow[rru, dashed] \arrow[rr] & & 0 \end{tikzcd}$$

to obtain a weak equivalence $\mathcal{R} \xrightarrow{\simeq} \mathcal{O}$. In particular, this is a weak equivalence between bifibrant objects. Therefore,

$$\begin{aligned} \iota_* Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O},\mathcal{O}) &\simeq \iota_* Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{R},\mathcal{R}) \\ &\simeq \iota_* Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\iota^*\mathcal{P},\mathcal{R}) \\ &\simeq Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\iota_*\mathcal{R}) \\ &\simeq Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\mathcal{P}), \end{aligned}$$

where in the last step we used that $\iota_*\iota^*(\mathcal{P}) \cong \mathcal{P}$ and that hence $\iota_*\mathcal{R} \leftarrow \iota_*\iota^*\mathcal{P}$ is a weak equivalence between fibrant objects. $\square$

Since ι_* is symmetric monoidal, we get an induced $\mathbb{E}_2$-algebra structure on $Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\mathcal{P})$, and in view of theorem 5.46, the above argument shows that if $U \subseteq X$ is affine, then

$$R\Gamma_U(\mathfrak{z}(\tilde{\mathcal{O}}_X)) \simeq R\Gamma_U(Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\mathcal{P}))$$

as $\mathbb{E}_2$-algebras. It now suffices to show that

$$R\Gamma_U(Q\mathcal{H}om_{\iota_*(\mathcal{O}_X\otimes\mathcal{O}_X)}(\mathcal{P},\mathcal{P})) \simeq \mathrm{Hom}_{A\otimes A}(P,P)$$

for $U = \mathrm{Spec}(A)$ and $P \overset{\simeq}{\twoheadrightarrow} A$ a cofibrant replacement. Since we will solely work with the affine open site from now on, we will denote the restriction of a presheaf $\mathcal{F}$ on $\mathrm{Open}(X)$ to a presheaf on $\mathrm{Aff}(X)$ simply by $\mathcal{F}$ for the remainder of this section.

Definition 5.52. Let $\mathcal{D}_X$ be the site of affine opens on $X \times_k X$ of the form $W \times_k W$ for $W \subseteq X$ affine open. Of course, this site is isomorphic to the affine open site on X, but it better conceptualizes sheaves coming from A-bimodules.

Lemma 5.53. *1. The maps*

$$\begin{aligned} \Delta_* &: \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k) \to \mathrm{Ch}((\hat{\mathcal{D}}_X)_k), \quad \mathcal{F} \mapsto (W \times_k W \mapsto \mathcal{F}(\Delta{-}1(W \times_k W)) = \mathcal{F}(W)) \text{ and} \\ \Delta^{-1} &: \mathrm{Ch}((\hat{\mathcal{D}}_X)_k) \to \mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k), \quad \mathcal{G} \mapsto (U \mapsto \mathrm{colim}_{\Delta(U)\subseteq W\times_k W}\, \mathcal{G}(W \times_K W) \cong \mathcal{G}(U \times_k U)) \end{aligned}$$

form an isomorphism of categories.

2. We have $\Delta_*(\mathcal{O}_X \otimes \mathcal{O}_X) \cong \mathcal{O}_{X\times_k X}$ and $\Delta^{-1}(\mathcal{O}_{X\times_k X}) \cong \mathcal{O}_X \otimes \mathcal{O}_X$. In particular, the above isomorphism yields an isomorphism

$$\mathrm{LMod}_{\mathcal{O}_{X\times_k X}}(\mathrm{Ch}((\hat{\mathcal{D}}_X)_k)) \cong \mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)).$$

3. *Both Δ_* and Δ^{-1} preserve all three classes of fibrations, cofibrations and weak equivalences in the presheaf categories as well as the left module categories.*

4. *For any commutative k-algebra A, the adjunction $\tilde{(-)} \vdash \Gamma_{\mathrm{Spec}(A)}$ between complexes of A-modules and complexes of presheaves of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules is a Quillen adjunction. In addition, $\tilde{(-)}$ preserves acyclic fibrations.*

5. *The previous statement remains true if we consider $\tilde{(-)}$ as a functor from complexes of $A \otimes_k A$-modules to complexes of presheaves of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules on the site $\mathcal{D}_{\mathrm{Spec}(A)}$.*

Proof. Statement 1. is true by construction of the functors.
For 2., simply note that $\mathcal{O}_{X\times_k X}(W \times_k W) \cong \mathcal{O}_X(W) \otimes_k \mathcal{O}_X(W) \cong (\mathcal{O}_X \otimes \mathcal{O}_X)(W)$.
For 3., note that we get two adjoint equivalences $\Delta^{-1} \dashv \Delta_*$ and $\Delta_* \dashv \Delta^{-1}$. Clearly both Δ^{-1} and Δ_* preserve acyclic fibrations, and hence both also preserve cofibrations. Since $\mathcal{D}_X$ is isomorphic as a site to $\mathrm{Aff}(X)$, the sheaf topos $(\tilde{\mathcal{D}}_X)_k$ also has enough points and hence we can check weak equivalences at stalks. But at the same time, the only points in the sheaf topos on $\mathcal{D}_X$ are of the form $\Delta(x)$ for $x \in X$, and $(\Delta_*\mathcal{F})_{\Delta(x)} \cong \mathcal{F}_x$ and $(\Delta^{-1}\mathcal{G})_x \cong \mathcal{G}_x$, proving that both Δ^{-1} and Δ_* preserve weak equivalencs.
For 4., first note that this is indeed an adjunction. To see this, let M be a complex of A-modules and consider a map $M \to \mathcal{F}(\mathrm{Spec}(A))$. If $U = \mathrm{Spec}(B)$ is an affine open of $X = \mathrm{Spec}(A)$, then we get a restriction map $\mathcal{F}(X) \to \mathcal{F}(\mathrm{Spec}(B))$ which is a map of A-modules. But $\mathcal{F}(\mathrm{Spec}(B))$ is a B-module, and hence we get a map $\mathcal{F}(X) \otimes_A B \to \mathcal{F}(\mathrm{Spec}(B))$. We can hence construct a map

$$\tilde{M}(\mathrm{Spec}(B)) \cong M \otimes_A B \to \mathcal{F}(X) \otimes_A B \to \mathcal{F}(\mathrm{Spec}(B)).$$

of B-modules. Now note that $\tilde{(-)}$ sends quasi-isomorphisms to pointwise weak equivalences: If $M \to N$ is a quasi-isomorphism of complexes of A-modules and $U = \mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$ is an affine open, then in particular B is flat over A and therefore $- \otimes_A B$ preserves quasi-isomorphisms. Hence $\tilde{(M)}(U) = M \otimes_A B \to N \otimes_A B = \tilde{N}(U)$ is again a quasi-isomorphism. Further, we already know that the global sections functor preserves acyclic fibrations. This shows that the above adjunction is Quillen. Now if $M \to N$ is an acyclic fibration, then so is $M \otimes_A B \to N \otimes_A B$. This finishes the proof.
For 5., just note that everything in the proof of 4. still works. $\square$

Proof of theorem 5.46. Let $\mathcal{P} \xrightarrow{\simeq} \mathcal{O}_X$ be a cofibrant resolution. The $\mathcal{O}_X \otimes \mathcal{O}_X$-module $\mathcal{P}$ is bifibrant and $\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(-,-)$ is a right Quillen bifunctor, implying that $Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})$ is again fibrant in $\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)$. We hence get a weak equivalence

$$R\Gamma_U(Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})) \simeq \mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})(U) \cong \mathrm{Hom}_{\mathcal{O}_U\otimes\mathcal{O}_U}(\mathcal{P}|_U,\mathcal{P}|_U)$$

We then have the following chain of weak equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_U\otimes\mathcal{O}_U}(\mathcal{P}|_U,\mathcal{P}|_U) &\cong \mathrm{Hom}_{\mathcal{O}_{U\times_k U}}((\Delta_U)_*(\mathcal{P}|_U),(\Delta_U)_*(\mathcal{P}|_U)) \\ &\cong \mathrm{Hom}_{\mathcal{O}_{X\times_k X}|_{U\times_k U}}(\Delta_*(\mathcal{P})|_{U\times_k U},\Delta_*(\mathcal{P})|_{U\times_k U}).\end{aligned}$$

By the above lemma $\Delta_*\mathcal{P}$ is again bifibrant, and $(\Delta_*\mathcal{P})|_{U\times_k U}$ is fibrant. We can therefore use proposition [Hin05, 1.7.3] with a choice of cofibrant resolution $\mathcal{P}' \xrightarrow{\simeq} (\Delta_*(\mathcal{P}))|_{U\times_k U}$ to get weak equivalences

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_{X\times_k X}|_{U\times_k U}}((\Delta)_*(\mathcal{P})|_{U\times_k U},(\Delta)_*(\mathcal{P})|_{U\times_k U}) &\xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{O}_{X\times_k X}|_{U\times_k U}}(\mathcal{P}',(\Delta)_*(\mathcal{P})|_{U\times_k U})\\ &\xleftarrow{\simeq} \operatorname{Hom}_{\mathcal{O}_{X\times_k X}|_{U\times_k U}}(\mathcal{P}',\mathcal{P}')\end{aligned}$$

Note that $(\Delta_*\mathcal{P})|_{U\times_k U} \xrightarrow{\simeq} (\Delta_U)_*\mathcal{O}_U \cong \tilde{A}$ is again a trivial fibration, and therefore $\mathcal{P}' \xrightarrow{\simeq} \tilde{A}$ is a cofibrant resolution in $\mathcal{O}_{U\times_k U}$-modules. Now let $P \xrightarrow{\simeq} A$ be a cofibrant resolution of A as an A^e-module. Then $\tilde{P}$ is a cofibrant $\mathcal{O}_{U\times_k U}$-module, and we hence get a weak equivalence $\tilde{P} \xrightarrow{\simeq} \mathcal{P}'$ between bifibrant objects. Therefore,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_{U\times_k U}}(\mathcal{P}',\mathcal{P}') &\xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{O}_{U\times_k U}}(\tilde{P},\mathcal{P}')\\ &\xleftarrow{\simeq} \operatorname{Hom}_{\mathcal{O}_{U\times_k U}}(\tilde{P},\tilde{P})\\ &\cong \operatorname{Hom}_{A\otimes A}(P,P).\end{aligned}$$

This proves that $R\Gamma_U(\mathfrak{z}(\tilde{\mathcal{O}}_X)(\mathfrak{a})) \simeq \mathfrak{z}(\tilde{A})(\mathfrak{a})$ as complexes of k-modules. Recall that $R\Gamma_U$ is lax symmetric monoidal, and therefore we get an induced evaluation map

$$Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(U)\otimes\mathcal{P}(U) \to R(Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(U)\otimes\mathcal{P}(U)) \to \mathcal{P}(U)$$

But $\mathcal{P}(U) \simeq \mathcal{O}_X(U) \cong A \simeq P$, so this is in fact equivalent to the evaluation map

$$\operatorname{Hom}_{A\otimes A}(P,P)\otimes P \to P$$

of the center of A. This shows that the above equivalence is indeed an equivalences of $\mathbb{E}_2$-algebras. □

5.3 Recovering polydifferential operators as the center of $\mathcal{O}_X$

Now suppose that X is also smooth. In this case, there is another definition of Hochschild cohomology developed by M. Kontsevich in [Kon03] given by the hypercohomology of the sheaf of polydifferential operators $\mathcal{D}_{\text{poly}}(X) \in \operatorname{Ch}(\operatorname{Sh}_k(X))$. If $U = \operatorname{Spec}(A) \subseteq X$ is an affine open, then

$$\begin{aligned}\mathcal{D}_{\text{poly}}(X)_n(U) &= \{f \in \operatorname{Hom}_k(A^{\otimes n},A) : f \text{ is a differential operator in each factor}\}\\ &\subseteq \operatorname{Hom}_k(A^{\otimes n},A) \simeq C^n(A,A).\end{aligned}$$

The sheaf of polydifferential operators inherits the structure of a homotopy Gerstenhaber algebra from the Hochschild cochain complex. It is quasi-coherent as an $\mathcal{O}_X$-module, and therefore fibrant in the local projective model structure on affine opens. We want to show that the sheaf of polydifferential operators is indeed a model of the center of $\mathcal{O}_X$, including the $\mathbb{E}_2$-algebra structure. We first make a comparison on the level of homotopy sheaves.

Theorem 5.54. *Let X be a smooth, quasi-compact, separated scheme over k. We have an equivalence*

$$Q\mathcal{D}_{\text{poly}}(X) \simeq \mathfrak{z}(\tilde{O}_X)(\mathfrak{a})$$

in the ∞-category $\operatorname{LMod}_{\tilde{\mathcal{O}}_X}(\operatorname{Sh}_\infty)$ of $\tilde{\mathcal{O}}_X$-modules.

It suffices to show this equivalence for the sites of affine opens, since they yield an equivalent ∞-category. Since $\iota_*\mathfrak{z}(\tilde{\mathcal{O}}_X)(\mathfrak{a}) \simeq Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})$, it suffices to show

$$\iota_*\mathcal{D}_{\mathrm{poly}}(X) \simeq \mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})$$

as presheaves of $\mathcal{O}_X$-modules. In the following we will suppress the restriction to affine opens.

Let $\mathcal{O}$ be an associative algebra in complexes of sheaves. If $\mathcal{F},\mathcal{G}$ are sheaves of left $\mathcal{O}$-modules, recall that $\mathbb{R}\mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{J})$ for some K-injective resolution $\mathcal{J}$ of $\mathcal{G}$ in the catgory of sheaves of left $\mathcal{O}$-modules. Let $\Delta : X \to X \times_k X$ be the diagonal. We have already used the adjunction $\Delta^{-1} \dashv \Delta_*$ induced by this in lemma 5.53 above, but we now want to consider the full site of affine opens on $X \times_k X$ instead of the smaller site $\mathcal{D}_X$, and we also consider sheaves instead of presheaves. In particular, the map $\Delta^{-1} : \mathrm{Ch}(\mathrm{Sh}(\mathrm{Aff}(X \times_k X))_k) \to \mathrm{Ch}(\mathrm{Sh}(\mathrm{Aff}(X))_k)$ is given by the presheaf version followed by sheafification. We then have $\Delta^{-1}\Delta_* \cong \mathrm{id}$ since X is separated. Denote by $\overset{a}{\otimes}$ the tensor product of sheaves.

Lemma 5.55. *1. We have a local quasi-isomorphism of complexes of presheaves*

$$\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X)$$

2. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\Delta_*\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\Delta^{-1}\mathcal{F},\mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_{X\times_k X}}(\mathcal{F},\Delta_*\mathcal{G})$.

3. If $\mathcal{O}_X \xrightarrow{\simeq} \mathcal{I}$ is a K-injective resolution in sheaves of $\mathcal{O}_X \overset{a}{\otimes} \mathcal{O}_X$-modules, then $\Delta_*\mathcal{O}_X \to \Delta_*\mathcal{I}$ is a K-injective resolution in $\mathcal{O}_{X\times_k X}$-modules.

Assuming this lemma, we can prove the theorem as follows.

Proof of theorem 5.54. By [Yek01, Corollary 2.9] we have a local weak equivalence

$$\Delta_*\mathcal{D}_{\mathrm{poly}}(X) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X\times_k X}}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X).$$

We then get the following chain of local weak equivalences

$$\begin{aligned}
\Delta_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X) &= \Delta_*\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{I}) \\
&\cong \Delta_*\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\Delta^{-1}\Delta_*\mathcal{O}_X,\mathcal{I}) \\
&\cong \mathcal{H}om_{\mathcal{O}_{X\times_k X}}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{I}) \\
&\simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X\times_k X}}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X) \\
&\simeq \Delta_*\mathcal{D}_{\mathrm{poly}}(X)
\end{aligned}$$

where in the second to last step we used 5.55(3.). Now note that Δ^{-1} preserves local weak equivalencs, and therefore

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X) \simeq \mathcal{D}_{\mathrm{poly}}(X).$$

Together with 5.53(1.) this finishes the proof. $\square$

Proof of lemma 5.55. For 1., let $\alpha : \mathcal{O}_X \otimes \mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_X \overset{a}{\otimes} \mathcal{O}_X$ be the unit of the sheafification adjunction. This is a weak equivalence of dg algebras in presheaves, and therfore induces a Quillen equialvalence

$$\mathrm{LMod}_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)) \underset{\alpha_*}{\overset{\alpha^*}{\underset{\perp}{\rightleftarrows}}} \mathrm{LMod}_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathrm{Ch}(\widehat{\mathrm{Aff}(X)}_k)) \ .$$

We therfore get the following chain of weak equivalences

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P}) &\simeq \mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\alpha_*\mathcal{O}_X)\\
&\cong \mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\alpha^*\mathcal{P},\mathcal{O}_X)\\
&\simeq \mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\alpha^*\mathcal{P},\mathcal{I})\\
&\cong \mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}((\alpha^*\mathcal{P})^a,\mathcal{I})\\
&\simeq \mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{I})\\
&= \mathbb{R}\mathcal{H}om_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X).
\end{aligned}$$

The 2. statement is standard.
For the 3. statement, note first that $\mathcal{O}_X$ and $\mathcal{I}$ are both fibrant in the local projective model structure, and the presheaf version of the $\Delta^{-1} \dashv \Delta_*$ adjunction is Quillen for this model structure on the affine open sites, so $\Delta_*\mathcal{O}_X \to \Delta_*\mathcal{I}$ is again a local weak equivalence. Further Δ^{-1} is exact, and therefore preserves acyclic complexes. Therefore, if $\mathcal{S}$ is an acyclic $\mathcal{O}_{X\times_k X}$-module, then

$$\mathrm{Hom}_{\mathcal{O}_{X\times_k X}}(\mathcal{S},\Delta_*\mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_X\overset{a}{\otimes}\mathcal{O}_X}(\Delta^{-1}\mathcal{S},\mathcal{I})$$

is acyclic, proving that $\Delta_*\mathcal{I}$ is K-injective. $\square$

Let $\mathcal{B}_\bullet(\mathcal{O}_X)$ denote the $\mathcal{O}_X\otimes\mathcal{O}_X$-module $U \mapsto B_\bullet(A)$. We have a surjective map $\mathcal{B}_\bullet(\mathcal{O}_X) \to \mathcal{O}_X$ given by multiplication. Hence for a projective resolution $\mathcal{P} \overset{\simeq}{\twoheadrightarrow} \mathcal{O}_X$ of $\mathcal{O}_X$ as an $\mathcal{O}_X \otimes \mathcal{O}_X$-module, we get a lift $\mathcal{P} \to \mathcal{B}_\bullet(\mathcal{O}_X)$. We get an evaluation map

$$\iota_*\mathcal{D}_{\mathrm{poly}}(X) \otimes \mathcal{B}_\bullet(\mathcal{O}_X) \to \iota_*\mathcal{O}_X$$

coming from the fact that $\mathcal{D}_{\mathrm{poly}}(X)$ is affine locally a subcomplex of the Hochschild complex. This lifts to a map

$$\begin{tikzcd}
Q\iota_*\mathcal{D}_{\mathrm{poly}}(X)\otimes\mathcal{P} \arrow[r, dashed] \arrow[d] & \mathcal{P} \arrow[d, two heads, "\simeq"] \\
\iota_*\mathcal{D}_{\mathrm{poly}}(X)\otimes\mathcal{B}_\bullet(\mathcal{O}_X) \arrow[r] & \mathcal{O}_X
\end{tikzcd}$$

and it is clear from the proof of theorem 5.54 that this map corresponds to the evaluation map

$$Q\mathcal{H}om_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{P},\mathcal{P})\otimes\mathcal{P} \to \mathcal{P}.$$

It therefore makes $Q\mathcal{D}_{\mathrm{poly}}(X)$ into a center of $\tilde{\mathcal{O}}_X$. In particular, this equips the sheaf of polydifferential operators with a new $\mathbb{E}_2$-algebra structure in the ∞-category of sheaves on X. Applying the derived U-sections functor for an affine open $U = \mathrm{Spec}(A)$, we obtain an $\mathbb{E}_2$-algebra structure on $\mathbb{R}\Gamma_U(Q\mathcal{D}_{\mathrm{poly}}(X)) \simeq D_{\mathrm{poly}}(A)$. As a final step, we want to compare this to the classical $\mathcal{G}er_\infty$-algebra structure.

5.4 The $\mathcal{G}er_\infty$-structures on an $\mathbb{E}_2$-algebra

Combining theorem 5.46 and 5.54, we see that for a smooth separated quasi-compact scheme X and an affine open $U = \mathrm{Spec}(A) \subseteq X$ we have a zig-zag of quasi-isomorphisms of $C_*(\mathbb{E}_2)$-algebras

$$\mathrm{D}_{\mathrm{poly}}(A) \simeq \mathrm{Hom}_{A\otimes A}(B_*(A), A),$$

and the later $C_*(\mathbb{E}_2)$-algebra structure recovers the classical Gerstenhaber algebra structure on homology. In particular, the $C_*(\mathbb{E}_2)$-algebra structure on $\mathrm{D}_{\mathrm{poly}}(A)$ also recovers the classical Gerstenhaber algebra structure in homology. To compare this to the classical $\mathcal{G}er_\infty$-algebra structure on polydifferential operators coming from the braces-algebra structure, we need to recall the construction of the zig-zag between the $\mathcal{G}er_\infty$-operad and the $C_*(\mathbb{E}_2)$-operad.

Proposition 5.56. *For any choice of a Drinfeld associator realized as an isomorphism of operads $D : \widehat{PaB}(k) \to \mathrm{Grp}(\widehat{PaCD})$, we have the following zig-zag of weak equivalences of dg operads*

$$C_*(\mathbb{E}_2) \to C_*(|\Pi_1\mathbb{E}_2|) \leftarrow C_*(|\widehat{PaB}(k)|) \xrightarrow{D} C_*(|\mathrm{Grp}(\widehat{PaCD})|) \to B_*U(\mathfrak{t}) \leftarrow \mathcal{G}er \leftarrow \mathcal{G}er_\infty.$$

The induced isomorphism on homology makes the following diagram commute

$$\begin{tikzcd} H_*(\mathcal{G}er_\infty) \arrow[r, "\cong"] & H_*(\mathbb{E}_2) \\ \mathcal{G}er \arrow[u, "\cong"] \arrow[r, "\mathrm{id}"'] & \mathcal{G}er \arrow[u, "\varphi_D"'] \end{tikzcd}$$

where φ_D sends the bracket to $\lambda \cdot [\gamma]$ for $D = (\lambda, \Phi)$.

Proof. The first part of the zig-zag is given by the adjunction $\Pi_1 \dashv |\cdot|$ between topological spaces and groupoids as described in [Pet14]. The fact that associators are precisely maps from $\widehat{PaB}(k)$ to $\mathrm{Grp}(\widehat{PaCD})$ is proved in [CiL24]. The map from $C_*(|\mathrm{Grp}(\widehat{PaCD})|)$ to the Bar complex of the completed universal enveloping algebra of the Drinfeld-Kohno Lie algebra is given by the map $PaCD \to U(\mathfrak{t})$ sending PaP to zero. The map $\mathcal{G}er \to B_*U(\mathfrak{t})$ is described in [Tam03]. In particular, all parts of the zig-zag are known explicitly, and to show that the square commutes it suffices to track the generating 2-ary operations $\cup$ and $[\cdot,\cdot]$ of $\mathcal{G}er_\infty$. The cofibrant replacement map $\mathcal{G}er_\infty \to \mathcal{G}er$ is the identity of 2-ary operations. The map $\mathcal{G}er \to B_*U(\mathfrak{t})$ sends the bracket to $t_{12} \in \mathfrak{t}_2$ and the product to $1 \in k$. The map $\mathrm{Grp}(\widehat{PaCD}) \to U(\mathfrak{t})$ by construction sends $H := t_{12} \cdot \mathrm{id}_{12}$ to t_{12}, and $1 \in B_0U(\mathfrak{t})$ corresponds to $12 \in C_0(\mathrm{Grp}(\widehat{PaCD}))$. On the other side of the zig-zag, the unit map of the fundamental groupoid adjunction sends $p \in C_0(\mathbb{E}_2(2))$ and $\gamma \in C_1(\mathbb{E}_2)$ to their respective equivalence classes, and $PaB \to \Pi_1\mathbb{E}_2$ hits p with the object 12 and $[\gamma]$ with the composition $\tilde{R} \circ R$ where R is the generating braid and $\tilde{R} = (21) \cdot R$. The problem is hence reduced to understanding the relationship between H and the image of $\tilde{R} \circ R$ under the associator. To this end, recall that if $X = 1 \cdot \tau_{12}$ then

$$D(R) = \exp\left(\frac{\lambda t_{12}}{2}\right) \cdot X.$$

Since D is an operad morphism, we also have $D(\tilde{R}) = \exp(\lambda t_{12}/2) \cdot \tilde{X}$ where $\tilde{X} = 1 \cdot \tau_{21}$. In particular,

$$D(\tilde{R} \circ R) = \exp(\lambda t_{12}) \cdot \mathrm{id}_{12}.$$

Finally note that the degree 1 part of this is given by $\lambda t_{12} \cdot \mathrm{id}_{12}$ since t_{12} has weight 1. This shows that the map

$$C_*(|\widehat{PaB}(k)|) \xrightarrow{D} C_*(|\mathrm{Grp}(\widehat{PaCD})|)$$

sends $\tilde{R} \circ R$ to λH, proving the proposition. $\square$

On the other hand, the classical $\mathcal{G}er_\infty$-structure on $D_{\mathrm{poly}}(A)$ is obtained from the Braces-algebra structure. In particular, Tamarkin proved in [Tam98] that for a choice of associator D we have a morphism of dg operads $\Psi_D : \mathcal{G}er_\infty \to$ Braces making the diagram

$$\begin{tikzcd} H_*(\mathcal{G}er_\infty) \arrow[r] & H_*(\mathrm{Braces}) \\ \mathcal{G}er \arrow[u, "\cong"] \arrow[r, "\mathrm{id}"'] & \mathcal{G}er \arrow[u, "\phi_D"'] \end{tikzcd}$$

commute where ϕ_D sends the bracket to Missing parts!

Now let $\mathcal{D} \in \mathrm{Alg}_{C_*(\mathbb{E}_2)}(\mathrm{Ch}(\mathrm{PSh}(X)))$ be a rectification of $Q\mathcal{D}_{\mathrm{poly}}(X) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sh}_\infty(X))$. Using [BM03, Theorem 4.4] and the above zig-zag, we obtain a $\mathcal{G}er_\infty$-algebra in the symmetric monoidal model category of complexes of presheaves; by abuse of notation denote this again by $\mathcal{D}$. In particular, we have a zig-zag of weak equivalences $\mathcal{D} \simeq \mathcal{D}_{\mathrm{poly}}(X)$ as complexes of presheavs. Without loss of generality, we may choose $\mathcal{D}$ to be fibrant. Then for $U = \mathrm{Spec}(A)$ affine open, $\mathbb{R}\Gamma(U, \mathcal{D}) \simeq \mathcal{D}(U)$ is a $\mathcal{G}er_\infty$-algebra in complexes, which by theorem 5.46 is quasi-isomorphic to the $\mathcal{G}er_\infty$-algebra structure induced by the same procedure on a rectification of the center of A. In particular,

$$\mathcal{D}(U) \simeq \mathcal{D}_{\mathrm{poly}}(A) \in \mathrm{Alg}_{\mathcal{G}er_\infty}(\mathrm{Ch}(k)),$$

where the $\mathcal{G}er_\infty$-algebra structure comes from the following.

Corollary 5.57. *Let A be a commutative regular k-algebra. Then by theorem 5.54, $Q\mathcal{D}_{\mathrm{poly}}(\mathrm{Spec}(A))$ admits an $\mathbb{E}_2$-algebra structure in the ∞-category of sheaves on $\mathrm{Spec}(A)$. Let $\mathcal{D} \in \mathrm{Alg}_{\mathcal{G}er_\infty}(\mathrm{Ch}(\mathrm{Sh}(\mathrm{Spec}(A))))^f$ be obtained by the above zig-zag of dg operads. Then*

$$\mathcal{D}(\mathrm{Spec}(A)) \simeq \mathcal{D}_{\mathrm{poly}}(\mathrm{Spec}(A))(\mathrm{Spec}(A)) = \mathcal{D}_{\mathrm{poly}}(A) \in \mathrm{Alg}_{\mathcal{G}er_\infty}(\mathrm{Ch}(k)).$$

The identity on $\mathcal{D}_{\mathrm{poly}}(A)$ lifts to an $\mathcal{G}er_\infty$-morphism between this $\mathcal{G}er_\infty$-structure and the one coming from $\mathcal{D}_{\mathrm{poly}}(A) \simeq \mathfrak{z}(\tilde{A})$. Both of these yield the classical Gerstenhaber algebra structure on $T_{\mathrm{poly}}(A)$.

5.5 Comparison to the classical homotopy Gerstenhaber algebra structure on polydifferential operators

For a smooth scheme X, the classical proofs of Deligne's conjecture equip $\mathcal{D}_{\mathrm{poly}}(X)$ with a $\mathcal{G}er_\infty$-algebra structure, and a Gerstenhaber algebra structure on hypercohomology. Unfortunately, we do not have enough information about the new $\mathcal{G}er_\infty$-algebra structure on $\mathcal{D}$ to get the following.

Conjecture 1. *Let $\mathcal{D}$ be a fibrant-cofibrant representative of the center $\mathcal{G}er_{infty}$-algebra structure for a smooth scheme X. Then the map*

$$\mathcal{D} \to \mathcal{D}_{\text{poly}}(X)$$

induced by the zig-zag of quasi-isomorphisms on the level of complexes of presheaves lifts to a $\mathcal{G}er_\infty$-quasi-isomorphism if we equip $\mathcal{D}_{\text{poly}}(X)$ with the classical $\mathcal{G}er_\infty$-algebra structure coming from Braces and Tamarkin's map.

Related to that, we would at least like the following statement, which we were not quite able to proof in the global case.

Conjecture 2. *The above quasi-isomorphism of $\mathcal{G}er_\infty$-algebras induces an isomorphisms of Gerstenhaber algebras on hypercohomology*

$$\mathbb{H}^*(X, \mathcal{D}) \to \mathbb{H}^*(\mathcal{D}_{\text{poly}}(X)).$$

This would enable us to use [CVdB10, Theorem 1.3] to get a formality result for this new $\mathcal{G}er_\infty$-algebra structure, and also access the action of the Grothendieck-Teichmueller group on formality morphisms developed in [DRW15].

While we cannot prove the global statement, [DP15, Corollary B.3] at least yields the statement in the case of an affine space.

Corollary 5.58. *Let X be an affine space. Then the identity map of $\mathcal{D}_{\text{poly}}(X)$ extends to a $\mathcal{G}er_\infty$-morphisms between the center $\mathcal{G}er_\infty$-algebra structure and the classical one.*

Proof. Consider the $\mathcal{G}er_\infty$-algebra structure on $T_{\text{poly}}(A)$ transferred via the HKR map from the center $\mathcal{G}er_\infty$-algebra structure on $\mathcal{D}_{\text{poly}}(A)$. By [DP15, Corollary B.3] and the fact that we get the correct Gernstenhaber algebra in cohomology, the identity on polyvector fields lifts to a $\mathcal{G}er_\infty$-morphism. We hence get a diagram of $\mathcal{G}er_\infty$-quasi-isomorphisms

$$\begin{tikzcd}
T_{\text{poly}}(A)^{\text{Shouten}} \arrow[r, rightsquigarrow, "\text{id}"] \arrow[d, rightsquigarrow, "\text{id}"'] & T_{\text{poly}}(A)^{\text{center}} \arrow[r, rightsquigarrow, "\text{HKR}"] & \mathcal{D}_{\text{poly}}(A)^{\text{center}} \\
T_{\text{poly}}(A)^{\text{Tamarkin}} \arrow[d, rightsquigarrow, "\text{HKR}"'] & & \\
\mathcal{D}_{\text{poly}}(A)^{\text{Tamarkin}} & &
\end{tikzcd}.$$

This proves the claim. $\square$

A The endomorphism ∞-category

We want to show that our endomorphism ∞-category $\mathcal{C}_{\mathfrak{a}}^{\otimes} \times_{\mathcal{M}} \mathcal{M}_{/M}$ agrees with Lurie's definition [Lur17, Definition 4.7.1.1]. This will show that our endomorphism ∞-category is the underlying category of a monoidal ∞-category.

Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. In particular, q exhibits $\mathcal{M} := \mathcal{C}_{\mathfrak{m}}$ as left-tensored over the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Construct an ∞-category $\mathcal{M}^{\circledast}$ as the fiber product

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} (N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1) \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\
N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1 \arrow[r, "\gamma"] & \mathcal{LM}^{\otimes}
\end{tikzcd} .$$

In particular, $\mathcal{M}^{\circledast}$ comes equipped with a coCartesian fibration $p : \mathcal{M}^{\circledast} \to N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1$. We have

$$\mathcal{M}^{\circledast}_{[0],0} = \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{[0], 0\} \simeq \mathcal{C}_{\mathfrak{m}} = \mathcal{M},$$

since the functor $\mathrm{LCut} : N(\mathbb{\Delta}^{\mathrm{op}}) \to \mathcal{LM}^{\otimes}$ sends $[0]$ to $(\langle 1 \rangle, \{1\}) = \mathfrak{m}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{M}^{\circledast} \times_{\Delta^1} \{1\} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} (N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1) \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\
N(\mathbb{\Delta}^{\mathrm{op}}) \times \{1\} \arrow[r, hook] \arrow[d] & N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1 \arrow[r, "\gamma"] \arrow[d] & \mathcal{LM}^{\otimes} \\
\{1\} \arrow[r, hook] & \Delta^1 &
\end{tikzcd} .$$

Then the upper right hand side square is a pullback by definition, the lower left hand side square is a pullback and the left hand side rectangle is a pullback, again by definition. By the pasting law, the upper left hand side square is a pullback, and hence, again by the pasting law, the large upper rectangle is a pullback. The lower horizontal arrow of this rectangle agrees with the map $\mathrm{Cut} : N(\mathbb{\Delta}^{\mathrm{op}}) \to \mathcal{LM}^{\otimes}$, so

$$\mathcal{M}^{\circledast} \times_{\Delta^1} \{1\} \simeq \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} N(\mathbb{\Delta}^{\mathrm{op}}).$$

But in the diagram

$$\begin{tikzcd}
\mathcal{C}_{\mathfrak{a}}^{\otimes} \times_{\mathrm{Assoc}^{\otimes}} N(\mathbb{\Delta}^{\mathrm{op}}) \arrow[r] \arrow[d] & \mathcal{C}_{\mathfrak{a}}^{\otimes} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\
N(\mathbb{\Delta}^{\mathrm{op}}) \arrow[r, "\mathrm{Cut}"] & \mathrm{Assoc}^{\otimes} \arrow[r, hook] & \mathcal{LM}^{\otimes}
\end{tikzcd}$$

both squares are pullbacks, so the rectangle is as well, and we get

$$\mathcal{M}^{\circledast} \times_{\Delta^1} \{1\} \simeq \mathcal{C}_{\mathfrak{a}}^{\otimes} \times_{\mathrm{Assoc}^{\otimes}} N(\mathbb{\Delta}^{\mathrm{op}}),$$

which is the $\mathbb{A}_{\infty}$-monoidal ∞-category corresponding to the monoidal ∞-catgory $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Call this $\mathbb{A}_{\infty}$-monoidal category $\mathcal{C}_{\mathfrak{a}}^{\circledast}$. Then p exhibits $\mathcal{M}$ as left-tensored over $\mathcal{C}_{\mathfrak{a}}^{\circledast}$ in the planar sense.

Proposition A.59. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Let $p : \mathcal{M}^{\circledast} \to N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1$ as above. Then the ∞-category $\mathcal{C}_{\mathfrak{a}}[M]$ from definition 2.5 is equivalent to the endormophism ∞-category of M as defined in [Lur17, Definition 4.7.1.1].*

Proof. Under $\gamma : N(\mathbb{\Delta}^{\mathrm{op}}) \times \Delta^1 \to \mathcal{LM}^{\otimes}$, the map

$$a : ([0], 0) \to ([1], 0)$$

sending the point in $[0]$ to $0 \in [1]$ maps to

$$\begin{aligned} \mathrm{LCut}(a) : (\langle 2 \rangle, \{2\}) &\to (\langle 1 \rangle, \{1\}) \\ 1 &\mapsto 1 \\ 2 &\mapsto 1. \end{aligned}$$

Interpreting $(\langle 2 \rangle, \{2\})$ as $(\mathfrak{a}, \mathfrak{m})$ and $(\langle 1 \rangle, \{1\})$ as $\mathfrak{m}$, this map corresponds to the unique element

$$\phi \in \mathrm{Mul}_{\mathcal{LM}}(\{\mathfrak{a}, \mathfrak{m}\}, \mathfrak{m}).$$

Similarly, the map

$$b : ([0], 0) \to ([1], 0)$$

sending the point in $[0]$ to $1 \in [1]$ maps to

$$\begin{aligned} \mathrm{LCut}(b) : (\langle 2 \rangle, \{2\}) &\to (\langle 1 \rangle, \{1\}) \\ 1 &\mapsto * \\ 2 &\mapsto 1. \end{aligned}$$

This map corresponds to the unique element

$$\psi \in \mathrm{Mul}_{\mathcal{LM}}(\{\mathfrak{m}\}, \mathfrak{m}).$$

Therefore, to give an enriched morphism of $\mathcal{M}$ is equivalent to giving a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

in $\mathcal{C}^{\otimes}$ such that

1. $q(\alpha) = \mathrm{LCut}(a)$,
2. $q(\beta) = \mathrm{LCut}(b)$, and
3. β is inert, i.e. q-coCartesian.

Unpacking this, M and N are objects in $\mathcal{M}$, and $X = (C, M')$ is an object in $\mathcal{C}^{\otimes}_{(\mathfrak{a},\mathfrak{m})} \simeq \mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{m}}$[1], while $\alpha : (C, M') \to M$ and $\beta : (C, M') \to N$ are morphisms in $\mathcal{C}^{\otimes}$ lifting ϕ and ψ respectively. Since q is coCartesian, there is a q-coCartesian lift for ϕ and $X = (C, M')$, namely the map $(C, M') \to C \otimes M'$. Hence, the data of α is equivalent to a map $C \otimes M' \to M$ in $\mathcal{M}$. Similarly, there is q-coCartesian

[1] This holds because of [Lur17, Proposition 2.1.2.12]

lift for ψ and $X = (C, M')$, namely the map $(C, M') \to M'$. Hence the data of β is equivalent to a map $M' \to N$ in $\mathcal{M}$, and since β is supposed to be q-coCartesian as well, this map has to be an equivalence. Hence, the ∞-category $\mathcal{C}_{\mathfrak{a}}[M]$ as defined in [Lur17, Definition 4.7.1.1] is equivalent to the ∞-category with objects given by pairs $(C \in \mathcal{C}_{\mathfrak{a}}, \eta : C \otimes M \to M)$, which is better known as

$$\mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/M}.$$

$\square$

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