Filtered quadratic algebras and PBW-Theorems

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MATH 780: Topics in Commutative Algebra

1 Introduction

The classical PBW-Theorem states that the associated graded algebra to the universal enveloping algebra of a finite dimensional Lie algebra is in fact simply the symmetric algebra on the underlying vector space. In this note we will explore the theory of A. Braverman and D. Gaitsgory aiming to explain this theorem in the context of the combinatorics of inhomogeneous quadratic algebras. We will also see some examples from algebraic geometry illustrating how the associated graded algebra is a form of linearization procedure.

2 Basic definitions

We start by defining filtered and graded objects, and explaining their relationship. Throughout these notes, let k be a field of characteristic zero. All k-algebras are assumed to be associative but not necessarily commutative unless stated otherwise. All tensor products are over k.

Definition 2.1. Let A be a k-algebra.

- A descending filtration on A is a family of submodules

$$A =: F_0A \supseteq F_1A \supseteq F_2A \supseteq \dots$$

  such that for all $i, j \geq 0$ we have $F_iA \cdot F_jA \subseteq F_{i+j}A$.

- An (exhaustive) ascending filtration on A is a family of submodules

$$F^0A \subseteq F^1A \subseteq \cdots \subseteq A$$

  such that $\bigcup_{i\geq 0} F^iA = A$ and for all $i, j \geq 0$ we have $F^iA \cdot F^jA \subseteq F^{i+j}A$.

A filtered algebra is a k-algebra A equipped with a descending or ascending filtration.

Example 2.2. (a) Let R be a commutative k-algebra and $I \trianglelefteq R$ an ideal. Then we have the I-adic (descending) filtration on R

$$R \supseteq I \supseteq I^2 \supseteq \dots .$$

(b) Let V be a finite dimensional k-vector space and $P \subseteq k \oplus V \oplus V^{\otimes 2}$ a subset. Consider the tensor algebra on V

$$T(V) := \bigoplus_{i \geq 0} V^{\otimes i}.$$

Then P generates an ideal $I = (P) \trianglelefteq T(V)$ and we have a projection

$$\pi : T(V) \to T(V)/I.$$

The k-algebra $A := T(V)/I$ inherits an (ascending) filtration by setting

$$F^i A := \pi \left(\bigoplus_{j \leq i} V^{\otimes j} \right).$$

Important cases of this situation are given by the universal enveloping algebra of a finite dimensional Lie algebra and the Clifford algebra of a quadratic form.

A descending filtered algebra carries a canonical topology such that the filtered pieces $F_i A$ are neighborhoods of $0 \in A$. This topology is Hausdorff if the filtration is separated, i.e. if $\varprojlim F_i A = 0$. In this case, we can use the filtration to talk about Cauchy sequences.

Definition 2.3. Let A be a k-algebra equipped with a separated descending filtration $\{F_i A\}_{i \geq 0}$. The completion of A is given by

$$\hat{A} := \varprojlim A/F_i A.$$

This is again equipped with a descending filtration

$$F_i \hat{A} := \ker \left(\hat{A} \to A/F_i \right) \cong \varprojlim_{j \geq i} F_i/F_j,$$

which is always separated since limits commute with limits. In particular, $\hat{A}$ is always Hausdorff and complete in the classical sense with respect to the above Cauchy sequences. We get a map

$$A \to \hat{A}$$

with kernel $\varprojlim F_i A$. We call A complete if this map is an isomorphism.

Remark 2.4. Another way to understand completeness of the topology induced by a filtration is the following: For each $i \geq 0$ we have a short exact sequence

$$0 \to F_i A \to A \to A/F_i A \to 0,$$

and we have an induced exact sequence

$$0 \to \varprojlim F_i A \to A \to \hat{A} \to {\varprojlim}^1 F_i A \to 0.$$

If A is complete, then the arrow $A \to \hat{A}$ is an isomorphism, and therefore both $\varprojlim F_i A$ and ${\varprojlim}^1 F_i A$ vanish.

Example 2.5. If we complete the integers with respect to the (p)-adic filtration for some prime $p \in \mathbb{Z}$, we get the p-adic integers $\hat{\mathbb{Z}}_p$.

Filtered objects carry a lot of extra structure, and can hence be difficult to work with. A simpler type of object is given by graded algebras.

Definition 2.6. A graded algebra is a k-algebra A equipped with a decomposition

$$A = \bigoplus_{i \geq 0} A_i$$

such that $A_i \cdot A_j \subseteq A_{i+j}$.

In particular, any graded algebra carries a "trivial" ascending filtration given by

$$F^i A = \bigoplus_{j \leq i} A_j.$$

Given an algebra with a descending filtration, we can form the associated graded algebra

$$\mathrm{gr}(A) := \bigoplus_{i \geq 0} F_i A / F_{i+1} A.$$

This is a much simpler object than the original filtered algebra, and in particular loses all topological information. To illustrate this, note that if the filtration is separated, we always have

$$\mathrm{gr}(\hat{A}) \cong \mathrm{gr}(A).$$

To understand in which way taking the associated graded simplifies the algebra structure, consider the multiplication on $\mathrm{gr}(A)$: If $\bar{x} \in \mathrm{gr}(A)_i$ and $\bar{y} \in \mathrm{gr}(A)_j$ with representatives $x \in F_i A$ and $y \in F_j A$ respectively, we have

$$\bar{x} \cdot \bar{y} = x \cdot y \mod F_{i+j+1} A.$$

This multiplication "cuts off" all of the higher order terms and only keeps the "leading term" of the original product. We may hence view the associated graded as a homogenization or linearization of the original filtered algebra.

Example 2.7. (a) Let X be a Noetherian scheme and $x \in X(k)$ a point. Then we have a local ring $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}_x$, and we take the $\mathfrak{m}_x$-adic topology on $\mathcal{O}_{X,x}$. The associated graded

$$\mathrm{gr}(\mathcal{O}_{X,x}) = \bigoplus_{i \geq 0} \mathfrak{m}_x^i / \mathfrak{m}_x^{i+1}$$

is the ring of global sections of the tangent cone $\mathrm{TC}_x(X)$ of X at x. Note that $\mathrm{gr}(\mathcal{O}_{X,x})_1 = \mathfrak{m}_x / \mathfrak{m}_x^2 = (\mathrm{T}_x X)^*$ generates this graded algebra, and we hence get a map

$$\mathrm{S}((T_x X)^*) \to \mathrm{gr}(\mathcal{O}_{X,x})$$

which yields an inclusion

$$\mathrm{TC}_x(X) \hookrightarrow \mathrm{T}_x X$$

of the tangent cone into the tangent space. If x is a regular point, then $\mathcal{O}_{X,x}$ is a domain, and therefore the $\mathfrak{m}_x$-adic topology is separated. In this case, the associated graded of $\mathcal{O}_{X,x}$ agrees with the associated graded of the completion, which by the Cohen Structure Theorem is isomorphic to a formal power series ring in $\dim T_x X$ variables. In particular, in this case the above map is an isomorphism.

(b) If $A = T(V)/I$ as before, we have

$$\mathrm{gr}(A) \cong T(V)/\mathrm{LH}(I),$$

where $\mathrm{LH}(I)$ is given by the leading homogeneous terms of elements in I. To see this, note that

$$F^iA/F^{i-1}A \cong (\bigoplus_{j\leq i} V^{\otimes j})/(\bigoplus_{j\leq i-1} V^{\otimes j} + (I \cap \bigoplus_{j\leq i} V^{\otimes j})).$$

3 Graded deformation theory

In the previous chapter we have seen that the multiplication in the associated graded of a filtered algebra differs from the original multiplication by "higher order terms". In this sense, the multiplication of the filtered algebra can be viewed as a perturbation of the multiplication in the graded algebra. We now want to make this precise.

Let A be a k-algebra with ascending filtration $\{F^iA\}_{i\geq 0}$. We can always construct a family of filtered k-algebras interpolating between A and its associated graded. To see this, let t be a formal parameter of degree 1 and define

$$\mathcal{A} := \bigoplus_{i\geq 0} F^iA \cdot t^i \subseteq A[t].$$

This is commonly called the Rees algebra of A, or Blow-up algebra in the context of algebraic geometry. Note that $\mathcal{A}$ is a graded $k[t]$-module, and $(F^iA \cdot t^i) \cdot (F^jA \cdot t^j) \subseteq F^{i+j} \cdot t^{i+j}$, so we indeed get a graded $k[t]$-algebra. In particular, we have an inclusion $k[t] \to \mathcal{A}$, and we will view $\mathcal{A}$ as a family over the affine line $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$. One immediately sees the following properties of the Rees algebra.

Lemma 3.8. *For each $\lambda \in k$, the quotient $\mathcal{A}/(t-\lambda)\mathcal{A}$ is a filtered k-algebra via the image of the filtration associated to $\mathcal{A}$ under the quotient map.*

- *If $\lambda = 0$, we get $\mathcal{A}/t\mathcal{A} \cong gr(A)$.*
- *If $\lambda \neq 0$, we get $\mathcal{A}/(t-\lambda)\mathcal{A} \cong A$.*

In particular, for all $\lambda \in k$ we have the same associated graded of the fiber over λ. We say the associated graded family of $\mathcal{A}$ is constant.

This construction is an example of a more general concept.

Definition 3.9. Let A be a graded algebra and t a formal parameter of degree 1.

- A graded deformation of order i of the algebra A is a graded $k[t]/t^{i+1}$-algebra structure on $A' := A \otimes k[t]/t^{i+1}$ such that $A'/tA' \cong A$ as graded algebras.
- A graded deformation of A is a graded $k[t]$-algebra structure on $A_t := A \otimes k[t]$ such that $A_t/t \cong A$ as graded algebras.

If A_t is a graded deformation of A, then as $k[t]$-modules $A_t \cong A[t]$, and if $\mu : A_t \otimes_{k[t]} A_t \to A_t$ is the multiplication of the $k[t]$-algebra structure, the restriction

$$A \otimes A \to A[t] \otimes_{k[t]} A[t] \xrightarrow{\mu} A[t]$$

is given by

$$a \otimes b \mapsto ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \dots$$

with $\mu_i : A \otimes A \to A$ of degree $-i$. This follows directly from the condition $A_t/t \cong A$. In particular, we see that we get back the original multiplication on A by discarding all "higher order terms" in t.

The condition that A_t be associative as an algebra imposes constraints on the maps μ_i. These conditions are called obstructions and just like in the non-graded case, they live in the Hochschild cohomology of the algebra.

Definition 3.10. Let A be a graded algebra and M a graded A-bimodule. The graded Hochschild cohomology of A with coefficients in M is given by

$$HH^*(A, M) := \mathrm{Ext}^*_{A \otimes A^{\mathrm{op}}}(A, M)$$

where the Ext is computed in the category of A-bimodules. This can be computed as the cohomology of the complex

$$C^i_{\mathrm{gr}}(A, M) := \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_k(A^{\otimes i}, M)_j$$

with $\mathrm{Hom}_k(A^{\otimes i}, M)_j$ the degree j maps, and the differential given analagously to the ungraded case.

Proposition 3.11 ([BG96], prop. 1.5)**.**

- *The set of isomorphism classes of first order deformations of A is isomorphic to $HH^2_{-1}(A, A)$.*
- *The obstruction to lifting a deformation of order i to a deformation of order $i + 1$ lies in $HH^3_{-i-1}(A, A)$*

Graded deformations have the following nice property, which follows from the same calculation as example 2.7(b).

Lemma 3.12. *Let A_t be a graded deformation of A. If $\lambda \in k$, the fiber $A_t/(t-\lambda)A_t$ over λ inherits the structure of a filtered algebra, and its associated graded*

$$gr(A_t/(\lambda - t)A_t) \cong A$$

is isomorphic to the original algebra A.

4 PBW-Theorems

We now return to example 2.2(b) and further analyze the structure of the associated graded. Again, let V be a finite dimensional k-vector space, and $P \subseteq k \oplus V \oplus V^{\otimes 2} = F^2(T(V))$ with $I = (P)$. In example 2.7(b) we have seen that the associated graded of the filtered algebra

$$A = T(V)/I$$

is isomorphic to the quotient

$$\mathrm{gr}(A) \cong T(V)/\mathrm{LH}(I)$$

by the leading homogeneous terms of the elements in I. But there is in fact a simpler way to produce a homogeneous quadratic algebra from P: Instead of first generating the full ideal and then taking leading homogeneous terms, one could instead just consider the leading terms of P itself.

Let $p : k \oplus V \oplus V^{\otimes 2} \to V^{\otimes 2}$ be the projection onto the homogeneous degree 2 component, and let $R = p(P)$ be the homogenization of P. Then

$$B := T(V)/(R)$$

is a graded algebra, and since $(R) \subseteq \mathrm{LH}(I)$ we get a canonical surjection

$$B = T(V)/(R) \to T(V)/\mathrm{LH}(I) \cong \mathrm{gr}(A).$$

Definition 4.13. We say that $A = T(V)/(P)$ satisfies the PBW-property with respect to P if this surjection is an isomorphism.

Example 4.14. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and consider the universal enveloping algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/(P)$$

with

$$P = \{v \otimes w - w \otimes v - [v, w] : v, w \in \mathfrak{g}\} \subseteq \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}.$$

Then

$$R = \{v \otimes w - w \otimes v : v, w \in \mathfrak{g}\} \subseteq \mathfrak{g}^{\otimes 2},$$

and therefore

$$T(\mathfrak{g})/(R) = \mathrm{S}(\mathfrak{g})$$

is the symmetric algebra on $\mathfrak{g}$. The PBW-property gets its name from the following classical theorem.

Theorem 4.15 (Classical PBW-Theorem)**.** *The symmetrization map*

$$S(\mathfrak{g}) \to U(\mathfrak{g})$$
$$x_1 \cdot \dots \cdot x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)}$$

induces an isomorphism of graded algebras

$$S(\mathfrak{g}) \cong U(\mathfrak{g}).$$

In other words, $U(\mathfrak{g})$ satisfies the PBW-property with respect to P as above.

Such PBW-theorems yield a very simple description of the associated graded. We would hence like to know

Question: How can we check whether $A = T(V)/I$ satisfies the PBW-property with respect to some P with $(P) = I$?

The answer to this question was, at least partially, given in A. Braverman and D. Gaitsgory's paper "Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type". The rest of this note will be concerned with explaining their strategy to this problem. First note

Lemma 4.16 ([BG96], Lemma 0.4)**.** *Suppose A satisfies the PBW-property with respect to P. Then*

(1) $P \cap F^1(T(V)) = 0$.

(2) $(F^1(T(V)) \cdot P \cdot F^1(T(V))) \cap F^2(T(V)) = P$.

Recall that a homogeneous quadratic algebra is called a Koszul algebra if its Koszul complex provides a projective resolution of A an A-bimodule. This is equivalent to the followig statement.

Lemma 4.17 ([BG96], Prop. A.2)**.** *A homogeneous quadratic algebra is Koszul if and only if for all graded A-bimodules M,*

$$HH^i_j(A, M) = 0 \quad \textit{if } i < -j.$$

We can now state the main theorem of Braverman-Gaitsgory

Theorem 4.18 ([BG96], Theorem 4.1)**.** *Let $A = T(V)/(P)$ and suppose that $B = T(V)/(R)$ is a Koszul algebra. Then A satisfies the PBW-property with respect to P if and only it P satisfies conditions (1) and (2) above.*

The idea behind the proof of this theorem is to find a graded deformation B_t of B such that the fiber over $1 \in k$ is given by A. Then by Lemma 3.12 we automatically get

$$B \cong \mathrm{gr}(B_t/(1-t)B_t) \cong \mathrm{gr}(A).$$

The reason for requiring that B be a Koszul algebra lies in the fact that these have very restricted Hochschild cohomlogy, and therefore fewer obstructions to lifting finite order deformations. In particular

Proposition 4.19 ([BG96], Prop. 3.7)**.** *Let A be a Koszul algebra.*

- *Let $i > 2$. Then for any deformation of A of order i there is at most one lift to an order $i+1$ deformation.*
- *If $i > 3$, there exists a lift of any order i deformation of A.*

Sketch of a proof for Theorem 4.18. If P satisfies condition (1), there exist maps $\alpha : R \to V$ and $\beta : R \to k$ such that

$$P = \{x - \alpha(x) - \beta(x) : x \in R\}.$$

We can identify α with a choice of first order multiplication map μ_1 and β with a choice of second order multiplication map μ_2. Condition (2) ensures that this μ_1 is a Hochschild 2-cocycle and that the first obstruction (involving μ_1 and μ_2) vanishes. This condition also implies that we can find a μ_3 such that the second obstruction vanishes. By proposition 4.19, there then exists a lift to a deformation over $k[t]$. □

Finally, we want to use theorem 4.18 to prove the classical PBW-Theorem.

Proof of the classical PBW-Theorem. Recall that we have $P = \{v \otimes w - w \otimes v - [v, w] : v, w \in \mathfrak{g}\}$ and $R = \{v \otimes ww \otimes v : v, w \in \mathfrak{g}\}$. In particular, $\alpha(v \otimes w - w \otimes v) = [v, w]$ and $\beta = 0$. Condition (1) states that P can not contain an element that lies in $F^1(T(V))$. The only element of P that would not intersect $\mathfrak{g}^{\otimes 2}$ is $[v, v]$ in the case $v = w$, and thus we see that this condition is equivalent to the antisymmetry of the Lie bracket.
Braverman-Gaitsgory show that condition (2) can be expressed in terms of the maps α and β. In particular, in the case $\beta = 0$, it is equivalent to

(i) $\mathrm{im}(\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha) \subseteq R$ on $R \otimes \mathfrak{g} \cap \mathfrak{g} \otimes R$,

(ii) $\alpha \circ (\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha) = 0$.

An element of $R \otimes \mathfrak{g} \cap \mathfrak{g} \otimes R$ is a linear combination of elements of the form

$$v = x \otimes y \otimes w - y \otimes x \otimes w - x \otimes w \otimes y + y \otimes w \otimes x + w \otimes x \otimes y - w \otimes y \otimes x,$$

and we have

$$\begin{aligned}(\alpha \otimes \mathrm{id} - \mathrm{id} \otimes \alpha)(v) &= [x, y] \otimes w + [w, x] \otimes y + [y, w] \otimes x - (x \otimes [y, w] + y \otimes [w, x] \otimes w \otimes [x, y]) \\ &= [x, y] \otimes w - w \otimes [x, y] + [w, x] \otimes y - y \otimes [w, x] + [y, w] \otimes x - x \otimes [y, w].\end{aligned}$$

This lies in R due to the Jacobi identity

$$[[x, y], w] + [[w, x], y] + [[y, w], x] = 0.$$

The same identity also ensures condition (ii), as

$$\begin{aligned}&\alpha([x, y] \otimes w - w \otimes [x, y] + [w, x] \otimes y - y \otimes [w, x] + [y, w] \otimes x - x \otimes [y, w]) \\ &\qquad = [[x, y], w] + [[w, x], y] + [[y, w], x].\end{aligned}$$

Since the symmetric algebra is indeed Koszul, this finishes the proof. □

References

[BG96] Alexander Braverman and Dennis Gaitsgory. Poincaré–Birkhoff–Witt theorem for quadratic algebras of Koszul type. *Journal of Algebra*, 181, 1996.