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Bachelorarbeit

Duality of Tensor Categories and Affine Supergroups and its Implications for Quantum Field Theory

angefertigt von

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1 Introduction

The objective of studying the representation theory of various algebraic structures most notably includes the possibility to recover information about the structure itself by examining its actions on better understood objects like modules or vector spaces. This gives rise to the notion of a representation, which is defined as a concrete realisation of an algebraic structure in form of a collection of operators on some space. Now the natural question arises as to what extend these representations determine the structure they represent, and in many cases the answer is that indeed all the information of the algebraic structure is encoded by its representation theory.

One of the easiest instances of this phenomenon is given by Pontryagin duality, which recovers an abelian locally compact topological group from the its group of characters. In 1938, Tadao Tannaka proved a similar theorem for compact but not necessarily abelian groups in his article "Über den Dualitätssatz der nichtkommutativen topologischen Gruppen" [Tan39], stating that a compact topological group can be recovered from its category of unitary representations. His proof was based on the forgetful functor sending an unitary representation to its underlying vector space, which is an additional structure of all representation categories. This theorem of Tannaka is the first of a whole class of results known as the *Tannaka reconstruction theorems*. These are generalizations of the original theorem to different algebraic structures, all stating that a monoid object A in a suitable category $\mathbf{C}$ can be recovered from the forgetful functor

$$F : A\mathbf{Mod} \to \mathbf{C}$$

over the module category of A.

In the first part of this bachelor thesis I will give an exposition of the respective theorem in the category of affine schemes over some field k. This particular Tannaka duality theorem recovering an affine group scheme from its category of finite dimensional representations was first studied by N. Saavedra Rivano in [SR72] and later proved by P. Deligne in [Del90]. In the later article, Deligne also proved a more general version of this reconstruction theorem in particular applying to affine super group schemes, which are the $\mathbb{Z}_2$-graded version of affine group schemes. In a later work, Deligne then went on to give a criterion classifying all categories that arise this way as the representation theory of some affine super group scheme [Del02]. A presentation of Deligne's proof of this result will occupy the second part of this thesis.

My motivation for investigating these two theorems stems from their significance in quantum field theory. In particular, I have been made aware of the possible application of Deligne's classification of representation categories of supergroups to the theory of supersymmetry in particle physics. By Wigner's conjecture stating that particle species of a quantum field theory correspond to the irreducible representations of its geometric symmetry group, one may use Tannaka-style reconstruction theorems to draw conclusions regarding these geometric symmetries from the knowledge of the particle species of the theory. One can then attempt to show that every sensible collection of particle species is precisely of the form required by Deligne's classification theorem, implying that supergroups form the most general symmetry structure of physical quantum field theories, and thus providing an intrinsic mathematical motivation for a supersymmetric extension of the standard model. This quest has been carried out in the third part of this article.

Notation. Unless stated otherwise, I will assume that algebras over some field or ring usually denoted by k are commutative and unital, and all mophisms of algebras are assumed to be unital. I will call structure preserving maps in a given category simply "morphism" instead of the often more conventional "homomorphism". Categories are denoted in bold font except for chapter 5, where tensor categories are denoted by calligraphic letters. The hom-sets of a category $\mathbf{C}$ are denoted by $\mathrm{Hom}_{\mathbf{C}}(-,-)$. In particular, I will use the following categories

Symbol	Objects	Morphisms
$\mathbf{Set}$	Sets	Functions
$\mathbf{Vect}_k$	Fin. dim. vector spaces over k	Linear maps
$\mathbf{Sch}$	Schemes	Homomorphisms of locally ringed spaces
$\mathbf{Alg}_k$	Algebras over k	Algebra homomorphisms
$\mathrm{R}\mathbf{Mod}$	(Left) modules over R	Module homomorphisms
$\mathbf{Comod}\mathcal{C}$	(Right) comodules over $\mathcal{C}$	Comodule homomorphisms
$\mathbf{Rep}(\mathrm{G})$	Fin. dim. lin. representations of G	G-equivariant linear maps
$G\mathbf{Set}$	G-sets	G-equivariant functions
$\mathbf{sVect}_k$	Fin. dim. super vector spaces over k	Even linear maps
$\mathbf{sAlg}_k$	Superalgebras over k	Even algebra homomorphisms
$\mathcal{H}\mathbf{ilb}$	Complex Hilbert spaces	Bounded linear maps
$\mathbf{Rep}_{\mathcal{H}}(G)$	Unitary representations of G on complex Hilbert spaces	Bounded G-equivariant linear maps
$\mathbf{C^{\star}Alg}_{\mathbb{C}}$	Unital C*-algebras over k	Unital $\star$-homomorphisms
$\mathbf{s}\mathcal{H}\mathbf{ilb}$	Complex super Hilbert spaces	Even bounded linear maps

Equivalence of categories is denoted by "$\sim$", and (not necessarily natural) isomorphisms are denoted by "$\simeq$".

2 Categorical preliminaries

Before we begin with the actual content of this thesis, I will give a brief survey on the categorical techniques used. In particular, since some of the definitions are inconsistent throughout the literature, I will clarify the notions used in the following sections.

2.1 Background

I will assume that the reader is familiar with basic category theory, in particular with the concept of limits and colimits as well as adjunctions. All of these category theoretical preliminaries can be found in Mac Lane's classical book [ML78]. I will further presume knowledge of commutative algebra and the theory of affine schemes.

The aim of this bachelor thesis is to classify categories which arise as linear representation categories of some kind of group. Therefore, the categories we will encounter are all designed to mimic the category of (finite dimensional) vector spaces over some field. This category has two major structures: its hom-spaces also have a vector space structure in a canonical way, and it is equipped with a tensor product and duals. These two structures can be generalized to abstract categories, yielding the notions of an abelian category and a monoidal category respectively.

2.2 Abelian categories

Definition 2.1. A category enriched over the category of abelian groups is called *abelian* if

- it has a zero object, i.e. an object which is initial and terminal,
- it has finite biproducts,
- every morphism has a kernel and a cokernel, and
- every monomorphisms is a kernel and every epimorphisms is a cokernel.

Let k be a field. An abelian category is said to be *k-linear* if all its hom-spaces are k-vector spaces, and composition is k-bilinear.

I will frequently write "direct sum" instead of "biproduct", since this is the usual name in categories related to the category of vector spaces over some field.

Definition 2.2. A functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories is called *additive* if all the maps $F : \mathrm{Hom}_{\mathbf{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{B}}(FX, FY)$ are morphisms of abelian groups. Analogously, a functor between k-linear categories is called k-linear if these maps are linear maps.

In an abelian category we have the notion of an exact sequence. In particular, a short exact sequence in an abelian category is a concatenation of morphisms

$$0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$$

such that a is monic, b is epic and $\mathrm{im}(a) = \ker(b)$. In this case X is a subobject of Y, and we have $Z \simeq Y/X$[1]. A morphism of two exact sequences $0 \to X \to Y \to Z \to 0$ and $0 \to X \to Y' \to Z \to 0$ is a morphism $Y \to Y'$ that restricts to the identity on X and induces the identity on Y. A short exact sequence as above is called *split* if it is isomorphic to the exact sequence

$$0 \to X \to X \oplus Z \to Z \to 0$$

with the morphisms given by the inclusion and projection coming from the biproduct diagram.

Definition 2.3. A non-zero object X of an abelian category is called *simple* if 0 and X are its only subobjects.

[1] If X is a subobject of Y, one denotes by Y/X the cokernel of the inclusion $X \to Y$ as a subobject.

This notion is well known from the representation theory of finite groups. There, a simple representation is usually called irreducible, and we have Schur's lemma stating that a morphism between irreducible representations is either zero or an isomorphism. A version of this lemma also exists in the more general setting.

Lemma 2.4 (Schur's lemma)**.** *If X and Y are two simple objects of an abelian category $\boldsymbol{C}$, then any non-zero morphism $X \to Y$ is an isomorphism. In particular, if X is simple then* $\mathrm{Hom}_{\boldsymbol{C}}(X, X)$ *is a division ring.*

Proof. If $f : X \to Y$ is a morphism, then the kernel of f is a subobject of X, and thus is all of X or zero. In the first case we get $f = 0$, and in the second case f is monic, so it implements X as a subobject of Y. Simplicity of Y then yields $X \simeq Y$ and f is an isomorphism. The second claim follows directly from the fact that addition and composition of endomorphisms make $\mathrm{Hom}_{\mathbf{C}}(X, X)$ into a (non-commutative) ring, and simplicity entails that every morphism has an inverse under composition. □

Corollary 2.5. *If $\boldsymbol{C}$ is abelian k-linear for an algebraically closed field k and every hom-space is finite dimensional over k, then for every simple object X we get* $\mathrm{Hom}_{\boldsymbol{C}}(X, X) = k$*. In particular, in this case all isomorphisms between simple objects are multiples of the identity.*

Proof. This follows directly from the above proposition and the fact that the only finite dimensional division algebra over an algebraically closed field k is k itself: If $\mathcal{D}$ is such a finite dimensional division algebra and $x \in \mathcal{D}$, then the inverse closed subring of $\mathcal{D}$ generated by x and k is an algebraic field extension over k, and hence equal to k. □

In abelian categories, we also have a generalization of the dimension of a vector space, called the *length* of an object.

Definition 2.6 (Finite length)**.** A Jordan–Hölder series of length n of an object X in an abelian category is a chain of inclusions as subobjects

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

such that X_i/X_{i-1} is simple for all $i = 1, \ldots, n$. Given a Jordan–Hölder series, we say that X contains Y with a multiplicity m if the number of values i for which $X_i/X_{i-1} \simeq Y$ is equal to m. An object is called of *finite length* if it admits a Jordan–Hölder series.

Jordan and Hölder showed that if an object admits a Jordan–Hölder series, then any such series admits every simple object with the same multiplicity, and in particular any two such sequences are of the same length. The length of an object is then defined as the length of one of its Jordan–Hölder series.

2.3 Monoidal categories

Definition 2.7 (Monoidal category)**.** A *monoidal category* is a quintuple $(\mathbf{C}, \otimes, a, I, l, r)$ where $\mathbf{C}$ is a category, $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a bifunctor called the *tensor product*, a is a natural isomorphism called the associativity constraint with components $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, I is an object of $\mathbf{C}$, and l and r are natural isomorphisms with components $l_X : I \otimes X \to X$ and $r_X : X \otimes I \to X$ respectively, called the unit constraints such that the following diagrams commute

$$
\begin{array}{ccccc}
 & & (W \otimes X) \otimes (Y \otimes Z) & & \\
 & \nearrow^{a_{W\otimes X,Y,Z}} & & \searrow^{a_{W,X,Y\otimes Z}} & \\
((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\
{\scriptstyle a_{W,X,Y}\otimes \mathrm{id}_Z}\downarrow & & & & \uparrow{\scriptstyle \mathrm{id}_W \otimes a_{X,Y,Z}} \\
(W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a_{W,X\otimes Y,Z}]{} & & W \otimes ((X \otimes Y) \otimes Z)
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\
 {\scriptstyle r_X \otimes \mathrm{id}_Y}\searrow & & \swarrow{\scriptstyle \mathrm{id}_X \otimes l_Y} \\
 & X \otimes Y &
\end{array}
$$

To form the 2-category of monoidal categories we also need a notion of a monoidal functor and monoidal natural transformation. Monoidal functors should preserve the tensor product structure, and thus the canonical definition is

Definition 2.8. A *lax monoidal functor* $F : (\mathbf{C}, \otimes_{\mathbf{C}}, a^{\mathbf{C}} I_{\mathbf{C}}, l^{\mathbf{C}}, r^{\mathbf{C}}) \to (\mathbf{D}, \otimes_{\mathbf{D}}, a^{\mathbf{D}}, I_{\mathbf{D}}, l^{\mathbf{D}}, r^{\mathbf{D}})$ between monoidal categories is a functor $F : \mathbf{C} \to \mathbf{D}$ together with a morphism $u : I_{\mathbf{D}} \to F(I_{\mathbf{C}})$ and a natural transformation μ with components $\mu_{X,Y} : F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$ such that the following diagrams commute.

$$
\begin{array}{ccc}
(F(X) \otimes_{\mathbf{D}} F(Y)) \otimes_{\mathbf{D}} F(Z) & \xrightarrow{a^{\mathbf{D}}_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathbf{D}} (F(Y) \otimes_{\mathbf{D}} F(Z)) \\
{\scriptstyle \mu_{X,Y} \otimes_{\mathbf{D}} \mathrm{id}_{F(X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes_{\mathbf{D}} \mu_{Y,Z}} \\
F(X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{D}} F(Z) & & F(X) \otimes_{\mathbf{D}} F(Y \otimes_{\mathbf{C}} Z) \\
{\scriptstyle \mu_{X \otimes_{\mathbf{C}} Y, Z}}\downarrow & & \downarrow{\scriptstyle \mu_{X, Y \otimes_{\mathbf{C}} Z}} \\
F((X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{C}} Z) & \xrightarrow[F(a^{\mathbf{C}}_{X,Y,Z})]{} & F(X \otimes_{\mathbf{C}} (Y \otimes_{\mathbf{C}} Z))
\end{array}
$$

$$
\begin{array}{ccc}
I_{\mathbf{D}} \otimes_{\mathbf{D}} F(X) & \xrightarrow{u \otimes_{\mathbf{D}} \mathrm{id}_{F(X)}} & F(I_{\mathbf{C}}) \otimes_{\mathbf{D}} F(X) \\
{\scriptstyle l^{\mathbf{D}}_{F(X)}}\downarrow & & \downarrow{\scriptstyle \mu_{I_{\mathbf{C}},X}} \\
F(X) & \xleftarrow[F(l^{\mathbf{C}}_X)]{} & F(I_{\mathbf{C}} \otimes_{\mathbf{C}} X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes_{\mathbf{D}} F(I_{\mathbf{C}}) & \xleftarrow{\mathrm{id}_{F(X)} \otimes_{\mathbf{D}} u} & F(X) \otimes_{\mathbf{D}} I_{\mathbf{D}} \\
{\scriptstyle \mu_{X,I_{\mathbf{C}}}}\downarrow & & \downarrow{\scriptstyle r^{\mathbf{D}}_{F(X)}} \\
F(I_{\mathbf{C}} \otimes_{\mathbf{C}} X) & \xrightarrow[F(r^{\mathbf{C}}_X)]{} & F(X)
\end{array}
$$

A lax monoidal functor is called *strong* if u and μ are both isomorphisms.

A monoidal natural transformation between monoidal functors (F, μ^F, u_F) and (G, μ^G, u_G) is a natural transformation $\alpha : F \to G$ such that the following diagrams commute.

$$
\begin{array}{ccc}
F(X) \otimes_{\mathbf{D}} F(Y) & \xrightarrow{\alpha_X \otimes_{\mathbf{D}} \alpha_Y} & G(X) \otimes_{\mathbf{D}} G(Y) \\
{\scriptstyle \mu^F_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mu^G_{X,Y}} \\
F(X \otimes_{\mathbf{C}} Y) & \xrightarrow[\alpha_{X \otimes_{\mathbf{C}} Y}]{} & G(X \otimes_{\mathbf{C}} Y)
\end{array}
$$

$$
\begin{array}{ccc}
 & I_{\mathbf{D}} & \\
{\scriptstyle u_F}\swarrow & & \searrow{\scriptstyle u_G} \\
F(I_{\mathbf{C}}) & \xrightarrow[\alpha_{I_{\mathbf{C}}}]{} & G(I_{\mathbf{C}})
\end{array}
$$

A monoidal category is called *strict* if the natural isomorphisms a, l and r are all equal to the identity natural transformations. By Mac Lane's coherence theorem, every monoidal category is strongly equivalent to a strict one, so one can usually suppress the associativity and unit constraints.

In many of the well known monoidal categories there is a way to "swap" two objects in a tensor product. For example in the category of finite dimensional vector spaces we have the linear map $V \otimes_k W \to W \otimes_k V$, $v \otimes w \mapsto w \otimes v$. The general notion of this is a *braiding*.

Definition 2.9. A *braided monoidal category* is a monoidal category equipped with a natural isomorphism τ with components $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ making the following diagrams commute.

$$\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\tau_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle \tau_{X,Y} \otimes \mathrm{id}_Z}\downarrow & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{id}_Y \otimes \tau_{X,Z}]{} & Y \otimes (Z \otimes X)
\end{array}$$

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{id}_X \otimes \tau_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle a^{-1}_{Z,X,Y}} \\
X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Y,Z}]{} & (X \otimes Z) \otimes Y & \xrightarrow[\tau_{X,Z} \otimes \mathrm{id}_Y]{} & (Z \otimes X) \otimes Y
\end{array}$$

A braided monoidal category is called *symmetric* if $\tau^{-1}_{X,Y} = \tau_{Y,X}$ for all objects X and Y.

Of course we want a functor between braided monoidal categories to respect the braiding structure. In particular,

Definition 2.10. A braided monoidal functor between braided monoidal categories is a monoidal functor (F, μ, u) making the following diagram commute

$$\begin{array}{ccc}
F(X) \otimes_{\mathbf{D}} F(Y) & \xrightarrow{\tau^{\mathbf{D}}_{F(X),F(Y)}} & F(Y) \otimes_{\mathbf{D}} F(X) \\
{\scriptstyle \mu_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mu_{Y,X}} \\
F(X \otimes_{\mathbf{C}} Y) & \xrightarrow{F(\tau^{\mathbf{C}}_{X,Y})} & F(Y \otimes_{\mathbf{C}} X)
\end{array}$$

A braided monoidal functor between symmetric monoidal categories is not required to satisfy additional properties. Also, no extra conditions are required of braided monoidal or symmetric monoidal natural transformations. In perspective of Mac Lane's coherence theorem, I will usually not explicitly write down the associativity or unit constraints of (symmetric, braided) monoidal categories.

Now that we have defined the notion of a tensor product, we can define dual objects. Let X be an object of some monoidal category $\mathbf{C}$.

Definition 2.11 (Dual object)**.** An object $X^\vee$ of $\mathbf{C}$ is said to be a *right dual* of X if there exist morphisms

$$\mathrm{ev} : X^\vee \otimes X \to I, \quad \mathrm{coev} : I \to X \otimes X^\vee$$

called the *evaluation* and *coevaluation*, such that the compositions

$$X \xrightarrow{\simeq} I \otimes X \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_X} (X \otimes X^\vee) \otimes X \xrightarrow{\simeq} X \otimes (X^\vee \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}} X \otimes I \xrightarrow{\simeq} X, \quad \text{and} \tag{1}$$

$$X^\vee \xrightarrow{\simeq} X^\vee \otimes I \xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{coev}} X^\vee \otimes (X \otimes X^\vee) \xrightarrow{\simeq} (X^\vee \otimes X) \otimes X^\vee \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_{X^\vee}} I \otimes X^\vee \xrightarrow{\simeq} X^\vee \tag{2}$$

are the identity morphism of X and $X^\vee$ respectively. A *left dual* ${}^\vee X$ is an object of $\mathbf{C}$ together with morphisms $\mathrm{ev} : X \otimes {}^\vee X \to I$ and $\mathrm{coev} : I \to {}^\vee X \otimes X$ satisfying the analogous condition.

Clearly, the unit object I is self-dual with evaluation and coevaluation given by the isomorphism $I \otimes I \simeq I$. Also, in a bradied monoidal category, a right dual is always a left dual and the other way around: If $\mathrm{ev} : X^\vee \otimes X \to I$ and $\mathrm{coev} : I \to X \otimes X^\vee$ make $X^\vee$ a right dual of X, then $\mathrm{ev} \circ \tau_{X,X^\vee} : X \otimes X^\vee \to I$ and $\tau^{-1}_{X^\vee,X} \circ \mathrm{coev} : I \to X^\vee \otimes X$ make $X^\vee$ a left dual of X. I will in this case simply speak of "the dual" of X.

Lemma 2.12. *If X is a dulaizable object of a monoidal category with zero object, then $X^{\otimes n} = 0$ for some $n \geqslant 0$ implies that $X = 0$.*

Proof. I will suppress all associativity and unit isomorphisms. We may assume that $n \geqslant 2$. Tensoring the composition

$$X \simeq I \otimes X \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_X} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}} X \otimes I \simeq X$$

with $X^{\otimes(n-2)}$, we obtain $\mathrm{id}_{X^{\otimes(n-1)}}$ at the one hand by the definition of a dual, and on the other hand

$$X^{\otimes(n-1)} \simeq I \otimes X^{\otimes(n-1)} \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_{X^{\otimes(n-1)}}} X^{\otimes n} \otimes X^\vee \xrightarrow{\mathrm{id}_{X^{\otimes(n-1)}} \otimes \mathrm{ev}} X^{\otimes(n-1)} \otimes I \simeq X^{\otimes(n-1)}.$$

But since $X^{\otimes n} = 0$, this shows that the identity of $X^{\otimes(n-1)}$ factors trough zero, and must therefore be the zero morphism. But then $X^{\otimes(n-1)} = 0$, and the result follows by induction. □

If $f : X \to Y$ is a morphism between dualizable objects X and Y, we have a dual morphism $f^\vee : Y^\vee \to X^\vee$ given by the composition

$$Y^\vee \simeq Y^\vee \otimes I \xrightarrow{\mathrm{id}_{Y^\vee} \otimes \mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id}_{Y^\vee} \otimes f \otimes \mathrm{id}_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^\vee}} I \otimes X^\vee \simeq X^\vee.$$

Definition 2.13. A monoidal category is called *rigid* if every object has a left and right dual.

As dualizability is a property intrinsic to a monoidal category, it is preserved by monoidal functors.

Proposition 2.14. *If $(F, \mu, u) : \boldsymbol{C} \to \boldsymbol{D}$ is a strong monoidal functor and X is a (right) dualizable object in $\boldsymbol{C}$ with right dual $X^\vee$, then $F(X)$ is (right) dualizable in $\boldsymbol{D}$ with (right) dual $F(X^\vee)$.*

Proof. The two morphisms

$$F(X^\vee) \otimes_{\mathbf{D}} F(X) \xrightarrow{\mu_{X^\vee,X}} F(X^\vee \otimes_{\mathbf{C}} X) \xrightarrow{F(\mathrm{ev}_X)} F(I_{\mathbf{C}}) \xrightarrow{u^{-1}} I_{\mathbf{D}} \quad \text{and}$$

$$I_{\mathbf{D}} \xrightarrow{u} F(I_{\mathbf{C}}) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes_{\mathbf{C}} X^\vee) \xrightarrow{\mu^{-1}_{X,X^\vee}} F(X) \otimes_{\mathbf{D}} F(X^\vee)$$

provide an evaluation and coevaluation making $F(X^\vee)$ the dual of $F(X)$ by functorality of F and the duality of $X^\vee$ and X. □

The internal hom functor. In the category of sets we have the notion of *currying*: A function from a set Z into the set of functions from X to Y is the same as a function from the set $Z \times X$ to Y. This works because the collection of morphisms between two sets is again a set. The generalization of this concept is called *internal hom*.

Definition 2.15. Let $\mathbf{C}$ be a symmetric monoidal category. An *internal hom* of $\mathbf{C}$ is a functor $[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ such that for every object X of $\mathbf{C}$ we have an adjunction

$$(- \otimes X) \dashv [X, -],$$

i.e. there are isomorphisms

$$\mathrm{Hom}_{\mathbf{C}}(Y \otimes X, Z) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, [X, Z])$$

natural in Y and Z. If $\mathbf{C}$ admits such a functor, it is said to be *closed*.

Remark 2.16. In a closed monoidal category the tensor product preserves colimits in the first variable, since all left adjoint functors preserve colimits. Dualy, the internal hom preserves limits in the second variable, as all right adjoint functors do.

Since the internal hom should be an object of the category behaving like the collection of morphisms between two objects, we expect to have an evaluation map. For objects X and Y of a closed monoidal category $\mathbf{C}$, the evaluation map

$$\mathrm{eval}_{X,Y} : [X, Y] \otimes X \to Y$$

is defined as the adjoint of the identity $\mathrm{id}_{[X,Y]}$.

In the case of a rigid symmetric monoidal category, we can always construct an internal hom by

$$[X, Y] := Y \otimes X^\vee.$$

Lemma 2.17. *The above defined functor $Y \rightsquigarrow Y \otimes X^\vee$ is indeed an internal hom. Further, we also get an adjunction*

$$(X^\vee \otimes -) \dashv (X \otimes -)$$

for each object X.

Proof. For the first part, it suffices to find a natural isomorphism with components

$$\mathrm{Hom}_{\mathbf{C}}(Y \otimes X, Z) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, Z \otimes X^\vee).$$

Such an isomorphism is given by $f \mapsto (f \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_Y \otimes \mathrm{coev})$ with inverse $g \mapsto (\mathrm{id}_Z \otimes \mathrm{ev}) \circ (g \otimes \mathrm{id}_X)$. That these are in fact inverse to each other follows directly from the identities (1) and (2). The isomorphisms for the second adjunction can be constructed very similarly. Send a morphism $f : X^\vee \otimes Y \to Z$ to the composition $(\mathrm{id}_X \otimes f) \circ (\mathrm{coev} \otimes \mathrm{id}_Y)$, and a morphism $g : Y \to X \otimes Z$ to the composition $(\mathrm{ev} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^\vee} \otimes g)$. That these are inverse to each other follows again from the duality identities. □

Since $(X^\vee)^\vee = X$, we have chains of adjunctions

$$\begin{aligned} &(- \otimes X) \dashv (- \otimes X^\vee) \dashv (- \otimes X), \quad \text{and} \\ &(X \otimes -) \dashv (X^\vee \otimes -) \dashv (X \otimes -). \end{aligned}$$

Therefore, in the case of a rigid symmetric monoidal category, the tensor product preserves all limits and colimits in both variables.

Trace and dimension. Let $\mathbf{C}$ be a symmetric monoidal category and X a dualizable object with dual $X^\vee$. By lemma 2.17 we have isomorphisms

$$\mathrm{Hom}_{\mathbf{C}}(Y \otimes X, Z) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, Z \otimes X^\vee)$$

for all objects Y and Z, and in particular taking $Y = I$ and $Z = X$ we get an isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(X, X) \simeq \mathrm{Hom}_{\mathbf{C}}(I \otimes X, X) \simeq \mathrm{Hom}_{\mathbf{C}}(I, X \otimes X^\vee).$$

By the proof of lemma 2.17, this isomorphism sends an endomorphism $f : X \to X$ to the morphism $(f \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X : I \to X \otimes X^\vee$.

Definition 2.18 (Trace and dimension)**.** The *trace* of an endomorphism $f : X \to X$ of a dualizable object X is given by

$$\mathrm{tr}(f) = (I \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \simeq X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} I) \in \mathrm{End}_{\mathbf{C}}(I).$$

The *dimension* of a dualizable object X is the trace of its identity morphism

$$\dim(X) = \mathrm{tr}(\mathrm{id}_X) = (\mathrm{ev}_X \circ \tau_{X,X^\vee} \circ \mathrm{coev}_X).$$

Proposition 2.14 implies that strong monoidal functors preserve traces and dimensions. The interaction between tensor product and trace is described by the next proposition and can be found in [PS14, cor. 5.10].

Proposition 2.19. *If $\mathbf{C}$ is a symmetric monoidal category and X and Y are dualizable objects, then for two endomorphisms $f : X \to X$ and $g : Y \to Y$ we have*

$$\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \circ \mathrm{tr}(g).$$

Corollary 2.20. *If X and Y are dualizable objects of a symmetric monoidal category, we have*

$$\dim(X \otimes Y) = \mathrm{tr}(\mathrm{id}_X \otimes \mathrm{id}_Y) = \mathrm{tr}(\mathrm{id}_X) \circ \mathrm{tr}(\mathrm{id}_Y) = \dim(X) \circ \dim(Y).$$

2.4 Internalization

Categorical internalization is the process of transferring an algebraic structure that is typically given by a set with some additional structure to an object of any category that possesses the required operations.

The most simple instance of this process is the notion of a *monoid object* in a monoidal category.

Definition 2.21. Let $\mathbf{C}$ be a monoidal category. A *monoid internal to $\mathbf{C}$* is an object M of $\mathbf{C}$ together with morphisms $m : M \otimes M \to M$ and $e : I \to M$ making the following diagrams commute.

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{\mathrm{id}_M \otimes m} & M \otimes M \\ {\scriptstyle m \otimes \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle m} \\ M \otimes M & \xrightarrow[m]{} & M \end{array} \qquad \begin{array}{ccccc} I \otimes M & \xrightarrow{e \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes e} & M \otimes I \\ & {\scriptstyle l_M}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle r_M} & \\ & & M & & \end{array}$$

A morphism of two internal monoids (M, m_M, e_M) and (N, m_N, e_N) in $\mathbf{C}$ is a morphism $f : M \to N$ in $\mathbf{C}$ such that the two diagrams

$$\begin{array}{ccc} M \otimes M & \xrightarrow{f \otimes f} & N \otimes N \\ {\scriptstyle m_M}\downarrow & & \downarrow{\scriptstyle m_N} \\ M & \xrightarrow[f]{} & N \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{e_M} & M \\ & {\scriptstyle e_N}\searrow & \downarrow{\scriptstyle f} \\ & & N \end{array}$$

commute.

For example, a monoid internal to the category of sets is a usual monoid, and a monoid internal to the category of abelian groups is a (non-commutative) unital ring. A monoid internal to the opposite category $\mathbf{C}^{\mathrm{op}}$ of some monoidal category $\mathbf{C}$ is called *comonoid internal to $\mathbf{C}$*. In particular, a comonoid internal to $\mathbf{C}$ is an object C in $\mathbf{C}$ together with morphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to I$ making the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta} \\ C \otimes C & \xrightarrow{\Delta \otimes \mathrm{id}_C} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} & & C & & \\ & \swarrow^{l_C^{-1}} & & \searrow^{r_C^{-1}} & \\ I \otimes C & \xleftarrow[\epsilon \otimes \mathrm{id}_C]{} & C \otimes C & \xrightarrow[\mathrm{id}_C \otimes \epsilon]{} & C \otimes I \end{array}$$

commute. Given the additional structure of a braiding, we can define what it means for an internal monoid or comonoid to be commutative.

Definition 2.22. Let **C** be a symmetric monoidal category. An internal monoid (M, m, e) of **C** is called *commutative* if

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\tau_{M,M}} & M \otimes M \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & M & \end{array}$$

commutes.

A internal cocommutative comonoid is defined analogously. A good example is the category of vector spaces, where an internal commutative monoid is just an algebra, and an internal commutative comonoid is a cogebra. One can build different structures by iterating internal monoids and comonoids; for example, a bialgebra is nothing but an internal commutative monoid in the category of internal cocommutative comonoids in $\mathbf{Vect}_k$, and in fact a commutative monoid internal to some category is just an internal monoid in the category of internal monoids of this category. We have an induced monoidal structure in the category of commutative monoids (or cocommutative comonoids) internal to some symmetric monoidal category **C**: For two such monoid objects (M, m_M, e_M) and (N, m_N, e_N), $(M \otimes N, m_M \otimes m_N, e_M \otimes e_N)$ is again an internal monoid, and the tensor unit has a monoid and comonoid structure given by the isomorphism $I \simeq I \otimes I$.

Proposition 2.23. *Lax monoidal functors send internal monoids to internal monoids.*

Proof. Let $(F, u, \mu) : (\mathbf{C}, \otimes_{\mathbf{C}}) \to (\mathbf{D}, \otimes_{\mathbf{D}})$ be a lax monidal functor and (A, m, e) a monoid in $(\mathbf{C}, \otimes_{\mathbf{C}})$. Then

$$I_{\mathbf{D}} \xrightarrow{u} F(I_{\mathbf{C}}) \xrightarrow{F(e)} F(A) \quad \text{and}$$
$$F(A) \otimes_{\mathbf{D}} F(A) \xrightarrow{\mu_{A,A}} F(A \otimes_{\mathbf{C}} A) \xrightarrow{F(m)} F(A)$$

are an unit and multiplication making $F(A)$ a monoid in $(\mathbf{D}, \otimes_{\mathbf{D}})$. □

The next step is defining modules over these internal monoids and comonoids.

Definition 2.24. Let **C** be a monoidal category and (M, m, e) an internal monoid in **C**. An *internal module over* M is an object X of **C** together with a morphism $\rho : M \otimes X \to X$ making the diagrams

$$\begin{array}{ccc} M \otimes M \otimes X & \xrightarrow{\mathrm{id}_M \otimes \rho} & M \otimes X \\ {\scriptstyle m \otimes \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \rho} \\ M \otimes X & \xrightarrow[\rho]{} & X \end{array} \qquad \begin{array}{ccc} I \otimes X & \xrightarrow{e \otimes \mathrm{id}_X} & M \otimes X \\ & {\scriptstyle l_X}\searrow & \downarrow{\scriptstyle \rho} \\ & & X \end{array}$$

commute. A morphism of internal M-modules (X, ρ) and (Y, σ) is a morphism $f : X \to Y$ such that

$$\begin{array}{ccc} M \otimes X & \xrightarrow{\rho} & X \\ {\scriptstyle \mathrm{id}_M \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ M \otimes Y & \xrightarrow[\sigma]{} & Y \end{array}$$

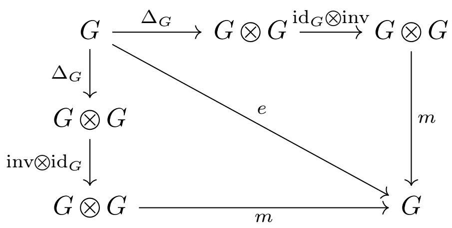
commutes.

The difference between a group and a monoid is the existence of inverses. To define what it means to be an inverse, one needs the structure of a diagonal map, i.e. a notion of "copying" an object, and thus monoidal categories are no longer sufficient to define internal groups.

Definition 2.25. A *cartesian category* **C** is a monoidal category whose monoidal structure is given by the categorical product and whose unit is a terminal object.

In a cartesian monoidal category, we have for every morphism $I \to X$ a unique morphism $X \to X$ given by precomposition by the unique morphism from X to the terminal object, and for each object X we have a diagonal morphism $\Delta_X : X \to X \otimes X$ induced by the identity morphism on X and the universal property of the product. If $f : I \to X$ is a morphism, I will also denote the unique induced morphism $X \to X$ by f. In this setting we can now define an internal group.

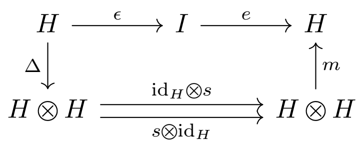
Definition 2.26. Let **C** be a cartesian category. A *group internal to* ***C*** is an object G of **C** together with morphisms $m : G \otimes G \to G$, $e : I \to G$ and $\mathrm{inv} : G \to G$ such that (G, m, e) is a monoid internal to the monoidal category **C** and



commutes.

A group object internal to the category of sets is again just a usual group, and a group internal to the category of topological spaces yields the concept of a topological group. In the next section I will work with group objects in the category of affine schemes as a cartesian category, and this will give the notion of an affine group scheme. Note that in a cartesian category, the unique morphism $X \to I$ and the diagonal map Δ_X make every object X into a commutative comonoid. Another important structure is that of a Hopf monoid.

Definition 2.27. Let **C** be a symmetric monoidal category. A *bimonoid internal to* ***C*** is an object of **C** equipped with a monoid and comonoid structure in a compatible way, i.e. the comultiplication and the counit are morphisms of monoids or the multiplication and the unit are morphisms of comonoids. A *Hopf monoid internal to* ***C*** is a bimonoid H together with a morphism $s : H \to H$ making the following diagram commutes.



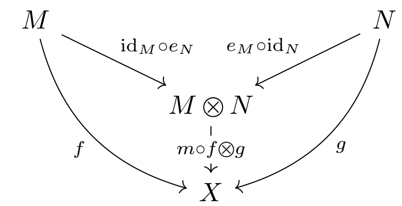
Note that a Hopf monoid in the opposite category is the same as a Hopf monoid in the original category. In a cartesian category, we can compare the notion of a Hopf monoid to that of a group in two different ways: We can choose the multiplication of the group to correspond to either the multiplication or the comultiplication of the Hopf monoid.

Lemma 2.28. *The tensor product in the category of commutative monoids internal to some symmetric monoidal category is the same as the coproduct of this category.*

Proof. Let (M, m_M, e_M) and (N, m_N, e_N) be two internal commutative monoids. We have morphisms

$$M \simeq M \otimes I \xrightarrow{\mathrm{id}_M \otimes e_N} M \otimes N \quad \text{and}$$
$$N \simeq I \otimes N \xrightarrow{e_M \otimes \mathrm{id}} M \otimes N$$

such that for every internal commutative monoid (X, m, e) and monoid morphisms $f : M \to X$, $g : N \to X$, the morphism $m \circ f \otimes g$ makes the following diagram commute.



This follows directly from the right hand side diagram in the definition of monoid morphism 2.21. It remains to show that this is the unique monoid morphism with this property. Suppose $h : M \otimes N \to X$ is another monoid morphism making the above diagram commute. The fact that h is a morphism of internal monoids is expressed in the commutativity of

$$\begin{array}{ccccc} M \otimes N \otimes M \otimes N & \xrightarrow{\simeq} & M \otimes M \otimes N \otimes N & \xrightarrow{m_M \otimes m_N} & M \otimes N \\ \downarrow{\scriptstyle h \otimes h} & & & & \downarrow{\scriptstyle h} \\ X \otimes X & & \xrightarrow[m]{\hspace{6em}} & & X \end{array}$$

Now precomposing this diagram with $\mathrm{id}_M \otimes e_N \otimes e_M \otimes \mathrm{id}_N$, the top arrow yields $\mathrm{id}_{M \otimes N}$ by the right hand side diagram in the definition of internal monoid 2.21, and the vertical left arrow becomes $f \otimes g$ by assumption. This shows $h = m \circ f \otimes g$. □

Proposition 2.29. *(a) A cocommutative Hopf monoid in a cartesian category* ***C*** *is the same as a group object in* ***C****.*

(b) A commutative Hopf monoid in a symmetric monoidal category ***C*** *is the same as a group object in the opposite category of the category of commutative monoids in* ***C****.*

Proof. "(a)" In a cartesian category, every object has a unique structure of a cocommutative comonoid with comultiplication the diagonal map and counit induces by the universal property of the terminal object, and every morphism is a morphism of comonoids. Therefore, a cocommutative bimonoid in **C** is the same as a monoid object. For any endomorphism of such a cocommutative bimonoid, the commutativity of the diagram in definition 2.27 is then equivalent to the commutativity of the diagram of definition 2.26 with the antipode s corresponding to the inversion morphism.
"(b)" By lemma 2.28, the tensor product in the category of commutative monoids in **C** is just the coproduct of this category. Therefore, its opposite category is cartesian, and we can use part (a) to state that a group object here is the same as a cocommutative Hopf monoid. But such a cocommutative Hopf monoid is clearly equivalent to a commutative Hopf monoid in **C**. □

A commutative Hopf monoid in the category of vector spaces recovers the usual notion of a Hopf algebra. I will enlarge upon the theory of commutative algebra of internal objects in section 5.4.

2.5 The ind-category

One often encounters the situation of working with a category of "finite" objects of some type, but still needing knowledge of how larger objects behave. For example, the tensor algebra of a finite dimensional vector space is no longer finite dimensional, since it is an infinite direct sum. In this case, one possible solution is to freely adjoin the required objects to the category in question. Denote by $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ the category of presheaves over a category $\mathbf{C}$.

Proposition 2.30. *Let $\boldsymbol{C}$ be a small category. The Yoneda embedding $y : \boldsymbol{C} \to [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ exhibits the presheaf category $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ as the free cocompletion of $\boldsymbol{C}$. In particular, for any cocomplete category $\boldsymbol{D}$ and functor $F : \boldsymbol{C} \to \boldsymbol{D}$ there is a functor $\tilde{F} : [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}] \to \boldsymbol{D}$ unique up to unique natural isomorphism such that $\tilde{F}$ preserves all colimits and $\tilde{F} \circ y \simeq F$.*

A proof of this statement can be found in [Dug98, prop. 2.2.4]. I will not be interested in all colimits, but only filtered colimits, i.e. colimits of diagrams indexed by small filtered categories.

Definition 2.31. Let $\mathbf{C}$ be a small category. An *ind-object* of $\mathbf{C}$ is a filtered colimit of objects in $\mathbf{C}$, where the colimit is taken in the cocompletion $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ of $\mathbf{C}$.

By construction, there exists an ind-object of $\mathbf{C}$ for every diagram indexed by a small filtered category, but these do not have to be objects of $\mathbf{C}$. In particular, taking the index category to be the one-object category $\mathbf{1}$, we see that every object X of $\mathbf{C}$ is an ind-object as colimit of the diagram $\mathbf{1} \to \mathbf{C}$ sending the object of $\mathbf{1}$ to X. More generaly, one can always identify an ind-object with the diagram in $\mathbf{C}$ giving the ind-object as its colimit.

To make the ind-objects of $\mathbf{C}$ into a category $\mathrm{Ind}(\mathbf{C})$, one has to define morphisms between them. We expect the inclusion $\mathbf{C} \to \mathrm{Ind}(\mathbf{C})$ described above to be fully faithful, and we also expect objects of $\mathbf{C}$ to be compact in $\mathrm{Ind}(\mathbf{C})$, i.e. for X in $\mathbf{C}$ the representable functor $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(X, -)$ should preserve filtered colimits. Therefore, for two ind-objects X and Y viewed as diagrams $X : \mathbf{D} \to \mathbf{C}$ and $Y : \mathbf{E} \to \mathbf{C}$ we must have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(X, Y) &= \mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(\operatorname*{colim}_{d \in \mathrm{ob}(\mathbf{D})} Xd, \operatorname*{colim}_{e \in \mathrm{ob}(\mathbf{E})} Ye) \\ &= \lim_{d \in \mathrm{ob}(\mathbf{D})} \mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(Xd, \operatorname*{colim}_{e \in \mathrm{ob}(\mathbf{E})} Ye) \\ &= \lim_{d \in \mathrm{ob}(\mathbf{D})} \operatorname*{colim}_{e \in \mathrm{ob}(\mathbf{E})} \mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(Xd, Ye) \\ &= \lim_{d \in \mathrm{ob}(\mathbf{D})} \operatorname*{colim}_{e \in \mathrm{ob}(\mathbf{E})} \mathrm{Hom}_{\mathbf{C}}(Xd, Ye). \end{aligned} \tag{3}$$

This already determines the morphisms between ind-objects uniquely, and we can therefore make the following definition.

Definition 2.32. The ind-category $\mathrm{Ind}(\mathbf{C})$ of some small category $\mathbf{C}$ has the ind-objects of $\mathbf{C}$ as its objects and morphisms according to equation (3).

If we have additional structure on $\mathbf{C}$, this structure may be transferred to $\mathrm{Ind}(\mathbf{C})$.

Proposition 2.33. *If $\boldsymbol{C}$ is small and abelian, then its ind-category $\mathrm{Ind}(\boldsymbol{C})$ also is an abelian category, and the embedding $\boldsymbol{C} \to \mathrm{Ind}(\boldsymbol{C})$ is exact. If $F : \boldsymbol{C} \to \boldsymbol{C}'$ is an additive functor of abelian categories, then the induced functor $\mathrm{Ind}(\boldsymbol{C}) \to \mathrm{Ind}(\boldsymbol{C}')$ is right (resp. left) exact if F is.*

For a proof of this statement see [KS06, section 8.6]. If $\mathbf{C}$ is abelian, it also holds that every object of $\mathrm{Ind}(\mathbf{C})$ is the filtered colimit of all its subobjects that lie in $\mathbf{C}$ [CB94].

Now suppose that $\mathbf{C}$ is rigid symmetric monoidal. Then in particular the tensor product is exact in both variables, and we obtain a tensor product on $\mathrm{Ind}(\mathbf{C})$ by

$$X \otimes Y = (\operatorname*{colim}_{d \in \mathrm{ob}(\mathbf{D})} Xd) \otimes (\operatorname*{colim}_{e \in \mathrm{ob}(\mathbf{E})} Ye) \simeq \operatorname*{colim}_{d,e \in \mathrm{ob}(\mathbf{C} \times \mathbf{D})} (Xd \otimes Ye).$$

By construction, the tensor unit I of $\mathbf{C}$ also is unital for the tensor product of $\mathrm{Ind}(\mathbf{C})$. Since we can check all diagrams in $\mathbf{C}$, this tensor product makes $\mathrm{Ind}(\mathbf{C})$ into a symmetric monoidal category. The rigid structure does not transfer to $\mathrm{Ind}(\mathbf{C})$ in this situation. Indeed, it is easy to see that an object of the ind-category is dualizable if and only if it already lies in $\mathbf{C}$: Suppose that X is dualizable in $\mathrm{Ind}(\mathbf{C})$ with dual $X^\vee$. Then since X is the filtered colimit of all its subobjects in $\mathbf{C}$, there must be a subobject X' of X lying in $\mathbf{C}$ such that the coevaluation $\mathrm{coev} : I \to X \otimes X^\vee$ factors through $X' \otimes X^\vee$. Then the commutative diagram

$$\begin{array}{ccccc} (X & \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_X} & X \otimes X^\vee \otimes X & \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}} & X) \quad = \mathrm{id}_X \\ & \searrow & \uparrow & & \uparrow \\ & & X' \otimes X^\vee \otimes X & \longrightarrow & X' \end{array}$$

shows that id_X factors through $\mathrm{id}_{X'}$, and thus we must have $X' = X$.

3 Affine group schemes

In this section I will state the basic properties of affine group schemes. These will be used in the following sections to prove the Tannaka duality theorem for affine groups.

3.1 Motivation

Suppose we are given a field k and a k-algebra R. For a free R-module M we can then look at the group of invertible transformations of unit determinant $\mathrm{SL}(M)$, which, upon choice of a basis, is given as matrix group $\mathrm{SL}_n(R)$ for $n = \dim M$. The condition of a matrix being in this group can be written as the vanishing of a polynomial in the entries of the matrix: If we have a matrix $A = (a_{ij})$ with entries in R, then this matrix is an element of $\mathrm{SL}_n(R)$ if and only if

$$\det(A) - 1 = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)} - 1 = 0.$$

Note however that this polynomial is independent from the k-algebra we chose. If we take $f \in k[x_{11}, \dots, x_{nn}]$ to be the polynomial above replacing a_{ij} by x_{ij}, we can see that for an arbitrary k-algebra S we have

$$\mathrm{SL}_n(S) = \mathrm{V}_S(f) := \{A = (a_{ij}) \in \mathrm{Mat}_n(S) : f(A) = 0\}.$$

Any matrix $(a_{ij}) \in \mathrm{Mat}_n(S)$ satisfying this condition therefore must be given by the images of the residue classes $\bar{x}_{ij}$ of the x_{ij} under some k-algebra homomorphism

$$k[x_{11}, \dots, x_{nn}]/(f) \to S.$$

In this sense we can view $(\bar{x}_{ij})$ as the *universal* matrix encoding the properties of the special linear group of any k-algebra.

This discussion shows that there is a natural generalization of usual groups of this type embracing the fact that their structure is independent of their concrete realization, and we call these generalized groups *affine group schemes*. In the case of the special linear group, all the information is encoded in the k-algebra $k[x_{11}, \dots, x_{nn}]/(f)$, which in turn corresponds to some affine subscheme of $\mathbb{A}^n_k$.

3.2 Definition and first properties

Before we can define affine group schemes we first need to investigate some properties of affine schemes. Let **Sch** be the category of schemes. By the Yoneda embedding, giving a scheme is equivalent to giving a contravariant representable functor from **Sch** to the category of sets, and we call this functor the functor of points. By looking at affine open covers, it is easy to show that a scheme is already uniquely determined by the restriction of its functor of points to the category of affine schemes (see [EH00, prop. 6.2]). Therefore, if we are given an affine scheme $X = \operatorname{Spec} R$, we have $\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S, X) \simeq \operatorname{Hom}_{\mathbf{Ring}}(R, S)$ and the functor of points of X is uniquely determined by the covariant functor h^R represented by R from **Ring** to the category of sets. The same construction works analogously for the category $\mathbf{Sch}/k$ of k-schemes, and the diagram below shows the setting we have now.

$$\mathbf{Sch}/k \xleftarrow{\operatorname{Spec}(-)} \mathbf{Alg}_k^{\mathrm{op}} \xrightarrow{h^{(-)}} \operatorname{Func}(\mathbf{Alg}_k, \mathbf{Set})$$

$$\mathbf{AffSch}/k \sim \mathbf{Alg}_k^{\mathrm{op}} \sim \operatorname{repFunc}(\mathbf{Alg}_k, \mathbf{Set})$$

Definition 3.1. An *affine group scheme* (over k) is a group object in the category of affine schemes over k. A morphism of affine group schemes is a scheme morphism that respects the multiplication law. Analogously, an affine monoid is an internal monoid in the category of affine schemes over k.

For the sake of brevity I will often use the name 'affine group' when meaning 'affine group scheme'. By the above considerations we immediately get the following.

Lemma 3.2. *The data of the following structures are essentially the same:*

(i) An affine group scheme,

(ii) A group object in $\mathbf{Alg}_k^{\mathrm{op}}$*. This is the same as a Hopf algebra over k (see proposition 2.29),*

(iii) A group object in the category $\mathbf{repFunc}(\mathbf{Alg}_k, \mathbf{Set})$*, which is a representable functor* $G : \mathbf{Alg}_k \to \mathbf{Set}$ *together with a natural transformation* $m : G \times G \to G$ *such that* $m(R)$ *is a group structure on* $G(R)$ *for all* k*-algebras* R*.*

If not stated otherwise, I will usually take the point of view of an affine group being a representable functor giving a group structure. The reason is that this way to think about affine groups makes it the easiest to define a representation theory of such groups. Also, the internal structures in the functor category $\mathbf{Func}(\mathbf{Alg}_k, \mathbf{Set})$ from k-algebras to sets is sometimes easier to handle than the explicit description in terms of prime ideals of certain rings.

The next piece of structure is the representing object of a given affine group. For an affine scheme $X = \operatorname{Spec} R$ we have a structure sheaf satisfying $\mathcal{O}_X(X) = R$, and I will adapt the notation used in the special case of affine varieties in calling $\mathcal{O}_X(X)$ the coordinate ring of X. For an affine group given by a representable functor, the object representing it is exactly the coordinate ring of the associated affine scheme. For an affine variety, the coordinate ring can be interpreted as k-algebra of local morphisms to the underlying affine line $\mathbb{A}^1_k$, and in the more general case of an affine scheme we can still define a type of evaluation of elements of the coordinate ring on points in the spectrum by defining the value of an element $f \in R$ on the point $\mathfrak{p} \in \operatorname{Spec} R$ to be the residue class of f under the projection morphism to $R/\mathfrak{p}$. It is therefore reasonable to expect the representing object of an affine group G to be given by the morphisms from G to some instance of the affine line.

Construction 3.3. As we have $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, we see that as a functor the affine line is represented by the polynomial ring in one variable. If R is another k-algebra, we have

$$\operatorname{Hom}_{\mathbf{Alg}_k}(k[x], R) \simeq R,$$

since every such homomorphism is uniquely determined by the image of x. Since the codomain of $\mathrm{Hom}_{\mathbf{Alg}_k}(k[x], -)$ is the category of sets, the image of a k-algebra R stays R as a set, but there is no k-algebra structure anymore. Therefore, the affine line is simply the forgetful functor of k-algebras. The transcendent variable x is the universal element mediating the natural transformation between these two view points.

Definition 3.4. The *coordinate ring* $\mathcal{O}(G)$ of an affine group G is given as a set by

$$\mathcal{O}(G) := \mathrm{Nat}(G, \mathbb{A}^1_k).$$

We make $\mathcal{O}(G)$ into a ring by defining

$$(f \pm g)_R := f_R \pm g_R$$
$$(f \cdot g)_R := f_R \cdot g_R$$

for $f, g \in \mathcal{O}(G)$ and R a k-algebra. Note that we utilized the ring structure on R on the right hand sides. We further make $\mathcal{O}(G)$ into a k-algebra by defining

$$(cf)_R := c \cdot f_R$$

for $c \in k$.

It remains to show that this is indeed a representing object for G.

Proposition 3.5. *Let G be an affine group with coordinate ring $\mathcal{O}(G) = \mathrm{Nat}(G, \mathbb{A}^1_k)$. Then the natural transformation*

$$\alpha : G \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), -)$$

sending an element $g \in G(R)$ to the k-algebra morphism "evaluation at g" is a natural isomorphism of set valued functors.

Proof. As an affine group, G is representable, so we know that $G = \mathrm{Hom}_{\mathbf{Alg}_k}(A, -)$ for some k-algebra A. By the Yoneda lemma we have an isomorphism of sets

$$\phi : \mathcal{O}(G) = \mathrm{Nat}(G, \mathbb{A}^1_k) \simeq \mathrm{Nat}(\mathrm{Hom}_{\mathbf{Alg}_k}(A, -), \mathbb{A}^1_k) \simeq \mathbb{A}^1_k(A) = A,$$

and by construction of the algebra structure on $\mathcal{O}(G)$, this even is an isomorphism of k-algebras. I claim that the α defined as above is just the induced natural isomorphism

$$\phi^* : \mathrm{Hom}_{\mathbf{Alg}_k}(A, -) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), -).$$

The component ϕ^*_R sends a homomorphism $g : A \to R$ to $g \circ \phi : \mathcal{O}(G) \to R$ which in turn maps an element f of the coordinate ring to $g(f_A(\mathbb{1}_A)) \in R$. The following commutative diagram coming from the naturality of f then proves the claim.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Alg}_k}(A, A) & \xrightarrow{f_A} & A \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Hom}_{\mathbf{Alg}_k}(A, R) & \xrightarrow[f_R]{} & R \end{array}$$

□

Definition 3.6. The element $a \in G(\mathcal{O}(G))$ corresponding to the natural transformation α via the Yoneda lemma is called the *universal element* of G.

Remark 3.7. The natural isomorphism α relates the two different ways to view $\mathcal{O}(G)$; as coordinate ring of the affine variety G and as representing object of the functor G. Take some $f \in \mathcal{O}(G)$. Then we get a morphism $f_R : G(R) \to R$ mapping $g \in G(R)$ to $f_R(g)$. But at the same time, using the evaluation morphism, g can be seen as a map $\mathrm{ev}_g : \mathcal{O}(G) \to R$, and we obtain the relation

$$f_R(g) = \mathrm{ev}_g(f).$$

Remark 3.8. We have a 'trivial' affine group $\star$ given by the functor sending every algebra to the one-element set viewed as the trivial group. The components of a natural transformation β from $\star$ to the affine line are given by an element $\beta_R \in R$ for all k-algebras R, and since we have a canonical inclusion $k \to R$ for every such algebra, naturality shows that β is already determined by $\beta_k \in k$. Therefore, the coordinate ring of the trivial affine group is just the underlying field

$$\mathcal{O}(\star) = k.$$

3.3 Affine groups as Hopf algebras

Recall from lemma 3.2 that an affine group may equivalently be seen as a Hopf algebra. I will now make this relation more explicit by showing that the group structure on a representable functor G indeed gives a Hopf algebra structure on the coordinate ring $\mathcal{O}(G)$.

The trick when transferring structure from the group to the coordinate ring and backwards is to once again use the isomorphism α from proposition 3.5.

Lemma 3.9. *Let $\Psi : G \to H$ a morphism of affine groups. Then there is a canonical morphism $\Psi^\sharp : \mathcal{O}(H) \to \mathcal{O}(G)$ such that*

$$(\Psi^\sharp f)_R(g) = f_R(\Psi_R g)$$

for all $f \in \mathcal{O}(H)$ and $g \in G(R)$.

Proof. Let α and β be the isomorphisms as in proposition 3.5 for G and H respectively. Then there is a unique morphism $\tilde{\Psi} : \mathrm{Hom}(\mathcal{O}(G), -) \to \mathrm{Hom}(\mathcal{O}(H), -)$ making the following square commute.

$$\begin{array}{ccc} G & \xrightarrow{\quad\Psi\quad} & H \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathrm{Hom}(\mathcal{O}(G), -) & \overset{}{\underset{\tilde{\Psi}}{\dashrightarrow}} & \mathrm{Hom}(\mathcal{O}(H), -) \end{array}$$

By the Yoneda lemma, $\tilde{\Psi}$ must be given by some morphism

$$\Psi^\sharp : \mathcal{O}(H) \to \mathcal{O}(G)$$

such that $\tilde{\Psi}_R = (\Psi^\sharp)^\star$ for all k-algebras R. For $g \in G(R)$, $f \in \mathcal{O}(H)$, commutativity of the above diagram then yields

$$f_R(\Psi_R g) = \mathrm{ev}_{\Psi_R g}(f) = \tilde{\Psi}(\mathrm{ev}_g)(f) = (\mathrm{ev}_g \circ \Psi^\sharp)(f) = \mathrm{ev}_g(\Psi^\sharp f) = (\Psi^\sharp f)_R(g).$$

□

We can now asses the Hopf algebra structure provided by an affine group. The multiplication map is a morphism

$$m : G \times G \to G,$$

and the tensor product in the category of k-algebras is indeed the coproduct of this category (see lemma 2.28). Thus, since the contravariant Yoneda embedding sends colimits to limits, and in particular coproducts to products, we have [2]

$$\mathrm{Hom}_{\mathbf{Alg}_k}(A_1 \otimes A_2, -) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(A_1, -) \times \mathrm{Hom}_{\mathbf{Alg}_k}(A_2, -),$$

and m can be viewed as a morphism

$$m : \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G) \otimes \mathcal{O}(G), -) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), -).$$

By the above lemma the induced morphism

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

then satisfies

$$(\Delta f)_R(g_1, g_2) = f_R(m_R(g_1, g_2)) =: f_R(g_1 \cdot g_2) \quad \forall g_1, g_2 \in G(R),$$

if we define $(f_1 \otimes f_2)_R(g, g') = (f_1)_R(g)(f_2)_R(g')$ for elements $f_1 \otimes f_2 \in \mathcal{O}(G) \otimes_k \mathcal{O}(G)$. By remark 3.8 the unit morphism $e : \star \to G$ yields a morphism $\epsilon : \mathcal{O}(G) \to k$ which by the lemma satisfies

$$(\epsilon f)_R(\star) = f_R(e_R \star).$$

We can state this less rigorously but more understandably as $\epsilon f = f(e)$, where e denotes the neutral object of the group $G(k)$. At last, the inversion morphism $\mathrm{inv} : G \to G$ yields an algebra morphism $S : \mathcal{O}(G) \to \mathcal{O}(G)$ satisfying

$$(Sf)_R(g) = f_R(\mathrm{inv}_R(g)) =: f_R(g^{-1}) \quad \forall g \in G(R).$$

Proposition 3.10. *Let G be a representable functor from k-algebras to sets, and let $m : G \times G \to G$, $e : \star \to G$ and $\mathrm{inv} : G \to G$ be natural transformations. Then (G, m, e, inv) is an affine group if and only if the morphisms Δ, ϵ and S defined as above make $\mathcal{O}(G)$ into a Hopf algebra.*

Proof. The functor $A \mapsto \mathrm{Hom}_{\mathbf{Alg}_k}(A, -)$ of k-algebras directly translates the diagrams of a Hopf algebra into those of an affine group. □

3.4 Representation theory for affine groups

Given an ordinary group G, a *linear representation* of G is defined as a group homomorphism

$$r : G \to \mathrm{Aut}(V)$$

for some vector space V. To generalize this notion to affine groups, we first need to think about a way to view the automorphism group as an affine group. To this end, let k be some commutative ring with unity, and let R be a k-algebra. Then for each k-module V we have the R-module obtained by extension of scalars to R,

$$V_R := V \otimes_k R.$$

We now define the endomorphism functor of V by

$$\underline{\mathrm{End}}(V) : \mathbf{Alg}_k \to \mathbf{Set}, \; R \mapsto \underline{\mathrm{End}}_R(V) := \mathrm{End}_{\mathbf{RMod}}(V_R).$$

The effect of this functor on morphisms is given by first restricting to V, then applying the morphism, and then extending to a module morphism again. In particular, if $\varphi : R \to S$ is a morphism of k-algebras,

$$\underline{\mathrm{End}}(V)(\varphi)(f)(v \otimes s) = s \cdot (\mathrm{id}_V \otimes \varphi)(f(v)).$$

[2] If $f : A_1 \to R$, $g : A_2 \to R$, write $[f, g]$ for the unique morphism $A_1 \otimes A_2 \to R$ given by f and g.

Remark 3.11. One can show that if V is finitely generated and projective, then $\underline{\text{End}}(V)$ indeed is an affine monoid under composition as multiplication (see [Nit04]). In particular, this is always the case if k is a field and we look at finite dimensional k-algebras. In this special case, upon a choice of basis, $\underline{\text{End}}_R(V)$ can be viewed as the $\dim V \times \dim V$ matrices with entries in R, and $\underline{\text{Aut}}(V) := \underline{\text{End}}(V)^\times$ is just the general linear group functor. We will mainly be concerned with the case of k being a field.

We are now able to talk about linear representations of affine groups.

Definition 3.12. Let G be an affine monoid. A linear representation of G on a k-module V is a natural transformation

$$r : G \to \underline{\text{End}}(V)$$

of monoid valued functors. If G is indeed an affine group, r will have values in $\text{Aut}(V)$.

Note that by the way we defined morphisms of affine groups, we could equally well say that r is a affine group homomorphism.

Remark 3.13. We can directly transfer most of the terminology used in ordinary representation theory. For example, we call a linear representation of an affine group G on V faithful if all morphisms r_R are injective, and we call a subspace W of V a subrepresentation of V, if $r_R(g)W_R \subset W_R$ for all k-algebras R and $g \in G(R)$.

Equivariant maps are also defined analogously to the case of ordinary groups.

Definition 3.14. A morphism of linear representations $u : (V, r) \to (V', r')$ (also called *intertwiner*) is a linear map $u : V \to V'$ such that the diagram

$$\begin{array}{ccc} V_R & \xrightarrow{u_R} & W'_R \\ {\scriptstyle r_R(g)}\downarrow & & \downarrow{\scriptstyle r'_R(g)} \\ V_R & \xrightarrow[u_R]{} & W'_R \end{array}$$

commutes for all k-algebras R and $g \in G(R)$, where u_R denotes the extension of scalars of u, i.e. $u_R = u \otimes \text{id}_R$.

The most important example of linear representation of affine groups is the regular representation. Recall that for an ordinary group G and $k[G]$ the k-module of functions from G to k (called the group algebra of G), the regular representation r_A of G on $k[G]$ is given by

$$r_A(g)(f)(h) = f(hg).$$

For an affine group G, the k-module of functions from G to k is nothing but the coordinate ring $\mathcal{O}(G) = \text{Nat}(G, \mathbb{A}^1)$, so we construct the regular representation of G on the k-vector space $\mathcal{O}(G)$. In particular, the regular representation r_A of G is a natural transformation with component at the k-algebra R

$$(r_A)_R : G(R) \to \text{Aut}_{\text{RMod}}(\mathcal{O}(G) \otimes_k R).$$

At this point, it is helpful to talk about the interpretation of $\mathcal{O}(G) \otimes_k R$.

Lemma 3.15. *The R-algebra $\mathcal{O}(G) \otimes_k R$ is the representing object of the affine scheme G_R over R given by restricting the functor G to the category of R-algebras. In particular, we have*

$$\mathcal{O}(G) \otimes_k R = \text{Nat}(G_R, \mathbb{A}^1_R) =: \mathcal{O}(G_R).$$

Proof. For every R-algebra S and k-algebra morphism $\phi : \mathcal{O}(G) \to S$ there is a unique R-linear extension of ϕ to $\mathcal{O}(G) \otimes_k R$. Thus, $\mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), S) \simeq \mathrm{Hom}_{\mathbf{Alg}_R}(\mathcal{O}(G) \otimes_k R, S)$, showing that $\mathcal{O}(G) \otimes_k R$ represents G_R. The proof of proposition 3.5 can be repeated in the same way replacing k with R, which proves the second claim. □

Remark 3.16. It is a central fact for the representation theory of affine groups that the representing object of the functor, the coordinate ring of the underlying affine scheme and the group algebra of the group all agree. We will see more of that when we examine the relation between representations of groups and comodules over the corresponding coordinate ring.

We are now able to finish the definition of the regular representation.

Construction 3.17. Let $g \in G(R)$, $f \in \mathcal{O}(G)$, and let $\varphi : R \to S$ be an R-algebra structure on the k-algebra S. Then for $x \in G(S)$, define

$$((r_A)_R(g)(f))_S(x) := f_S(x \cdot G(\varphi)(g)).$$

We can then uniquely extend this construction to $f \in \mathcal{O}(G) \otimes_k R = \mathrm{Nat}(G_R, \mathbb{A}^1_R)$, in particular,

$$((r_A)_R(g)(f \otimes r))_S(x) = \varphi(r) f_S(x \cdot G(\varphi)(g)).$$

There is a close connection between actions of an affine group and coactions of its group algebra. In particular, we have the following remarkable theorem

Theorem 3.18. *Let G be an affine monoid over k and V a k-module. There is a natural one-to-one correspondence between linear representations of G on V and $\mathcal{O}(G)$-comodule structures on V.*

In particular, if r is such a representation, the universal element $a \in G(\mathcal{O}(G))$ is mapped to an element of $\underline{\mathrm{End}}_{\mathcal{O}(G)}(V)$ whose restriction to V is a comodule structure, and conversely, if ρ is an $\mathcal{O}(G)$-comodule structure, then there is a unique representation r such that the restriction of $r_R(g)$ to V is given by $(\mathrm{id}_V \otimes ev_g) \circ \rho$ for all k-algebras R and $g \in G(R)$.

$$\begin{array}{ccccc} V & \hookrightarrow & V \otimes_k \mathcal{O}(G) & \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}(g)} & V \otimes_k R \\ & \searrow^{\rho} & \downarrow{\scriptstyle r_{\mathcal{O}(G)}(a)} & & \downarrow{\scriptstyle r_R(g)} \\ & & V \otimes_k \mathcal{O}(G) & \xrightarrow[\mathrm{id}_V \otimes \mathrm{ev}(g)]{} & V \otimes_k R \end{array}$$

Proof. First show that the above operations are indeed inverse to each other. Clearly, if ρ is a morphism of k-modules $V \to V \otimes_k \mathcal{O}(G)$, then $(\mathrm{id}_V \otimes \mathrm{ev}_a) \circ \rho = \rho$, as the map corresponding to a is the identity by definition. Now suppose we are given a natural transformation of set valued functors $r : G \to \underline{\mathrm{End}}(V)$. Then for $s \in R$, $g \in G(R)$ and $v \in V$,

$$s \cdot (\mathrm{id}_V \otimes \mathrm{ev}_g) \circ r_{\mathcal{O}(G)}(a)(v) = \underline{\mathrm{End}}(V)(\mathrm{ev}_g)(r_{\mathcal{O}(G)}(a))(v \otimes s) = r_R(g)(v \otimes s)$$

by naturality of r in $\mathrm{ev}_g : \mathcal{O}(G) \to R$.

Next show that r is indeed a natural transformation of group valued functors if and only if the corresponding ρ is a $\mathcal{O}(G)$-comodule structure. By definition, group multiplication in $\mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), R)$ is given by $m_R(f, g) = [f, g] \circ \Delta$. Therefore, for $g, h \in G(R)$, the action of $r_R(gh)$ on V is given by

$$V \xrightarrow{\rho} V \otimes_k \mathcal{O}(G) \xrightarrow{\mathrm{id}_V \otimes \Delta} V \otimes_k \mathcal{O}(G) \otimes_k \mathcal{O}(G) \xrightarrow{\mathrm{id}_V \otimes [\mathrm{ev}_g, \mathrm{ev}_h]} V \otimes_k R,$$

while the action of $r_R(g) r_R(h)$ is given by

$$V \xrightarrow{\rho} V \otimes_k \mathcal{O}(G) \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_h} V \otimes_k R \xrightarrow{\rho \otimes \mathrm{id}_R} V \otimes_k \mathcal{O}(G) \otimes_k R \xrightarrow{\mathrm{id}_V \otimes [\mathrm{ev}_g, \mathrm{id}_R]} V \otimes_k R,$$

which is the same as

$$V \xrightarrow{\rho} V \otimes_k \mathcal{O}(G) \xrightarrow{\rho \otimes \mathrm{id}_{\mathcal{O}(G)}} V \otimes_k \mathcal{O}(G) \otimes_k \mathcal{O}(G) \xrightarrow{\mathrm{id}_V \otimes [\mathrm{ev}_g, \mathrm{ev}_h]} V \otimes_k R.$$

It is now easy to see that $r_R(gh)$ and $r_R(g)r_R(h)$ agree for all k-algebras R and all $g, h \in G(R)$ if and only if the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes_k \mathcal{O}(G) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes_k \mathcal{O}(G) & \xrightarrow[\rho \otimes \mathrm{id}_{\mathcal{O}(G)}]{} & V \otimes_k \mathcal{O}(G) \otimes_k \mathcal{O}(G) \end{array}$$

At last, for the neutral element $e \in G(k)$ the identification $\mathrm{ev}_e = \epsilon$ with ϵ the counit of $\mathcal{O}(G)$ shows that $r_k(e) = \mathrm{id}_{V \otimes_k k}$ if and only if the following diagram commutes

$$\begin{array}{ccccc} V & \hookrightarrow & V \otimes_k \mathcal{O}(G) & \xrightarrow{\mathrm{id}_V \otimes \epsilon} & V \otimes_k k \\ & \searrow^{\rho} & & & \downarrow{\scriptstyle \mathrm{id}_{V \otimes_k k}} \\ & & V \otimes_k \mathcal{O}(G) & \xrightarrow[\mathrm{id}_V \otimes \epsilon]{} & V \otimes_k k \end{array}$$

Since this is exactly the counit diagram for V, this completes the proof. □

As an example, look again at the regular representation. An $\mathcal{O}(G)$-algebra structure on a k-algebra R is given by a morphism $\varphi : \mathcal{O}(G) \to R$, which is the same as giving a group element $h \in G(R)$ by $\varphi = \mathrm{ev}_h$. Then by construction 3.17 the morphism $(r_A)_{\mathcal{O}(G)}(a)$ acts on $f \in \mathcal{O}(G)$ by

$$((r_A)_{\mathcal{O}(G)}(a)(f))_R(g) = f_R(g \cdot G(\mathrm{ev}_h)(a)) = f_R(gh) = (\Delta f)_R(g, h),$$

since $G(\mathrm{ev}_g)(a) = g$ by naturality of α. Therefore, the regular representation of G corresponds to the $\mathcal{O}(G)$-comodule $\mathcal{O}(G)$ with coaction Δ.

The special role of the regular representation in the theory of affine groups is the same as in ordinary representation theory: It is a faithful representation which contains every other representation of the group. In particular, we have the following.

Proposition 3.19. *Let G be a flat affine monoid*[3] *over k, (V, r) a linear representation of G with associated $\mathcal{O}(G)$-comodule structure ρ, and*

$$V \otimes_k \mathcal{O}(G) \xrightarrow{\mathrm{id}_V \otimes \Delta} V \otimes_k \mathcal{O}(G) \otimes_k \mathcal{O}(G)$$

the free $\mathcal{O}(G)$-comodule structure on V. Then $\rho : V \to V \otimes_k \mathcal{O}(G)$ is an injective homomorphism of representations of G.

Proof. The coassociativity diagram making (V, ρ) an $\mathcal{O}(G)$-comodule is the same diagram stating that ρ is indeed a morphism of $\mathcal{O}(G)$-comodules. By the counit diagram of (V, ρ), the composition of ρ with $\mathrm{id}_V \otimes \epsilon$ is injective, so ρ must be too. □

[3]Meaning that the coordinate ring of G is flat as k-module.

3.5 Categorical formulation

Let $(\mathcal{C}, \Delta, \epsilon)$ be a cogebra over a ring k. Then the comodules over $\mathcal{C}$ together with morphisms of comodules form a additive category **Comod**$\mathcal{C}$. The forgetful functor to k-modules is exact if and only if $\mathcal{C}$ is flat, and further if this is the case, then the **Comod**$\mathcal{C}$ is abelian k-linear (in fact even a Grothendieck category) [Wis75].

A bialgebra structure (m, e) on $\mathcal{C}$ defines a monoidal structure on **Comod**$\mathcal{C}$. If (V, ρ) and (V', ρ') are comodules over $\mathcal{C}$, then on $V \otimes_k V'$

$$V \otimes_k V' \xrightarrow{\rho\otimes\rho'} V \otimes_k \mathcal{C} \otimes_k V' \otimes_k \mathcal{C} \xrightarrow{\simeq} V \otimes_k V' \otimes_k \mathcal{C} \otimes_k \mathcal{C} \xrightarrow{\mathrm{id}_V\otimes\mathrm{id}_{V'}\otimes m} V \otimes_k V' \otimes \mathcal{C}$$

is again a comodule structure; the tensor product of (V, ρ) and (V', ρ'). The unit $e : k \to \mathcal{C} \simeq k\otimes_k\mathcal{C}$ provides the tensor unit. In case k is a field, for any finite dimensional k-vector space V we get

$$\mathrm{Hom}_{\mathbf{kMod}}(V, V \otimes_k \mathcal{C}) \simeq \mathrm{Hom}_{\mathbf{kMod}}(V \otimes_k V^\vee, \mathcal{C}) \simeq \mathrm{Hom}_{\mathbf{kMod}}(V^\vee, \mathcal{C} \otimes_k V^\vee),$$

and thus a right coaction ρ on V yields a left coaction ρ' on $V^\vee$. A Hopf algebra structure S on $\mathcal{C}$ then provides a way to turn this into a right coaction again. In particular, the morphism

$$V^\vee \xrightarrow{\rho'} \mathcal{C} \otimes_k V^\vee \xrightarrow{\simeq} V^\vee \otimes_k \mathcal{C} \xrightarrow{\mathrm{id}_{V^\vee}\otimes S} V^\vee \otimes_k \mathcal{C}$$

is a right coaction on $V^\vee$, called the dual comodule structure of ρ. By construction, the forgetful functor preserves both the rigid and the monoidal structure.

Now let G be an affine monoid, and let $\mathbf{Rep}(G)$ be the category of finitely generated linear representations of G. Theorem 3.18 shows that this cateogory is equivalent to the category of finitely generated $\mathcal{O}(G)$-comodules, and therefore is a abelian k-linear with an exact forgetful functor in case G is flat. The equivalence between these two categories respects the monoidal and rigid structure, as can be checked by direct computation.

The flatness of a cogebra $\mathcal{C}$ also ensures that subcomodules over $\mathcal{C}$ are well defined. In particular, if (V, ρ) is a $\mathcal{C}$-comodule, and $W \subset V$ is a k-submodule, then we have $W \otimes_k \mathcal{C} \subset V \otimes_k \mathcal{C}$, and we say that W is a subcomodule of V if $\rho(W) \subset W \otimes_k \mathcal{C}$. If k is Noetherian and $\mathcal{C}$ is a flat cogebra, then every comodule over $\mathcal{C}$ is the filtered union of its finitely generated subcomodules [Ser93]. Therefore, in case that G is a flat affine monoid over a Noetherian ring, every linear representation of G is the union of its finitely generated subrepresentations.

4 Tannaka duality for affine group schemes

I will now prove the Tannaka duality theorem for affine group schemes, and also give a criterion for a category to be of the form $\mathbf{Rep}(G)$ for some affine group G. This approach is in analogy with the theory of Pontryagin duality, which describes how to recover a locally compact abelian group from its group of characters and also classifies the properties of the group in comparison to its dual group of characters.

4.1 General Tannaka duality

In a suitably well-behaved category, there is a strong connection between an internal monoid and its category of modules: One can always recover the monoid from only knowing the module category over it. In particular, if $\mathbf{C}$ is a locally small symmetric monoidal category with internal hom and all limits, and A is an internal monoid in $\mathbf{C}$, then A is isomorphic to the end[4] of the composition of the hom-functor with the forgetful functor $F : A\mathbf{Mod} \to \mathbf{C}$:

$$A \simeq \int_{M:A\mathbf{Mod}} \mathrm{Hom}_{\mathbf{C}}(F(M), F(M)).$$

This is just a restatement of the enriched Yoneda lemma and can be found in [Bor94, section 6]. Interestingly, in some categories it suffices to only look at the full subcategory of dualizable ("finite") module to recover the monoid, and in these cases one speaks of a *Tannaka reconstruction theorem.*

The easiest example is the cartesian category **Set** of sets, where one can recover a group from the category of sets equipped with an action of this group.

Theorem 4.1. *Let G be a group and write $G\mathbf{Set}$ for its category of G-sets. If $F : G\mathbf{Set} \to \mathbf{Set}$ is the forgetful functor forgetting the group action, then there is a canonical group isomorphism*

$$\mathrm{Aut}(F) \simeq G.$$

Proof. First observe that the set of morphisms from G viewed as a G-set to any G-set (X, ρ) is isomorphic to X: If $f : G \to X$ is such a morphisms, then

$$f(g) = f(ge) = \rho(g)f(e),$$

so one may only choose the image of e freely. Therefore, the forgetful functor F is represented by G, and we get

$$\mathrm{End}(F) \simeq \mathrm{End}(\mathrm{Hom}_{G\mathbf{Set}}(G, -)) \simeq \mathrm{Hom}_{G\mathbf{Set}}(G, G) \simeq G$$

using the Yoneda lemma. □

4.2 The reconstruction theorem

The aim of this section is to discuss the following theorem, which essentially establishes Tannaka duality between affine groups and their category of finite dimensional linear representations.

Theorem 4.2 (Reconstruction theorem)**.** *Let G be a flat affine monoid over a Noetherian ring k, and let R be a k-algebra. Suppose that for each linear representation (V, r_V) of G on a finitely generated k-module V we are given an R-linear morphism*

$$\lambda_V : V \otimes_k R \to V \otimes_k R$$

satisfying

(1) $\lambda_{V \otimes_k W} = \lambda_V \otimes \lambda_W$,

(2) $\lambda_k = id_R$[5], and

(3) for all G-equivariant linear maps $u : V \to W$ we have $\lambda_W \circ u_R = u_R \circ \lambda_V$.

Then there is a unique element $g \in G(R)$ such that $\lambda_V = (r_V)_R(g)$ for all finitely generated representations (V, r_V).

I will first present a proof of the theorem and then explore its consequences and corollaries. The proof relies heavily on the following lemma.

[4] For the definition of an end see [ML78, section 9.5].

[5] Where k denotes the trivial representation of G.

Lemma 4.3. *Let G be an affine monoid over k, and let u be a k-algebra endomorphism of its coordinate ring $\mathcal{O}(G)$. If the diagram*

$$\begin{array}{ccc} \mathcal{O}(G) & \xrightarrow{\Delta} & \mathcal{O}(G)\otimes_k \mathcal{O}(G) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{O}(G)}\otimes u} \\ \mathcal{O}(G) & \xrightarrow[\Delta]{} & \mathcal{O}(G)\otimes_k \mathcal{O}(G) \end{array}$$

commutes, then there is an element $g \in G(k)$ such that $u = (r_A)_k(g)$.

Proof. If $u^* : \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), -) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G), -)$ denotes the corresponding natural transformation, then $\Psi : G \to G$ given by $\Psi_R = \alpha_R^{-1} \circ u^* \circ \alpha_R$ satisfies

$$(uf)_R(g) = f_R(\Psi_R g)$$

for all k-algebras R and $g \in G(R)$ by lemma 3.9, with α being the natural transformation from proposition 3.5. Evaluating the two compositions of the diagram we get

$$(\Delta \circ u)(f)_R(x, y) = (\Delta(uf))_R(x, y) = (uf)_R(xy) = f_R(\Psi_R(xy))$$

and

$$\begin{aligned} ((\mathrm{id}_{\mathcal{O}(G)} \otimes u) \circ \Delta)(f)_R(x, y) &= \left((\mathrm{id}_{\mathcal{O}(G)} \otimes u) \left(\sum_i f_i \otimes g_i \right) \right)_R (x, y) = \left(\sum_i f_i \otimes ug_i \right)_R (x, y) \\ &= \sum_i (f_i)_R(x) \cdot \underbrace{(ug_i)_R(y)}_{(g_i)_R(\Psi_R y)} = (\Delta f)_R(x, \Psi_R y) = f_R(x \cdot (\Psi_R y)), \end{aligned}$$

where $\Delta f =: \sum_i f_i \otimes g_i$. By the commutativity of the diagram, this yields

$$f_R(\Psi_R(xy)) = f_R(x(\Psi_R y))$$

for all $f \in \mathcal{O}(G)$, and thus $\Psi_R(xy) = x(\Psi_R y)$ for all $x, y \in G(R)$.

Now let $e \in G(k)$ and $e_R \in G(R)$ be the respective neutral elements, and let $\iota_R : k \to R$ be the morphism making R a k-algebra. Then by naturality of Ψ, $G(\iota_R)$ sends $\Psi_k(e)$ to $\Psi_R(e_R)$, and we get

$$\begin{aligned} (uf)_R(x) &= f_R(\Psi_R x) = f_R(\Psi_R(x e_R)) = f_R(x(\Psi_R e_R)) = f_R(x(G(\iota)(\Psi_k e))) \\ &= ((r_A)_k(\Psi_k e)(f))_R(x), \end{aligned}$$

so defining $g := \Psi_k e$, we get the desired result. □

Using the lemma the proof of the reconstruction theorem is now straight forward.

Proof of theorem 4.2. Under the given assumptions every linear representation V of G is the union of its finite dimensional subrepresentations, $V = \bigcup_i V_i$. All the inclusion maps $V_i \to V_i \cap V_j$ are G-equivariant, and thus by condition (3) we can then glue all λ_{V_i} together to form a morphism $\lambda_V : V \otimes_k R \to V \otimes_k R$ also satisfying all the conditions.

Accordingly, we also have a morphism $\lambda_A : \mathcal{O}(G) \otimes_k R \to \mathcal{O}(G) \otimes_k R$ corresponding to the regular representation r_A of G. The multiplication on $\mathcal{O}(G)$ is equivariant for the representations r_A and $r_A \otimes r_A$, and thus (1) and (3) imply that λ_A is indeed a morphism of k-algebras. Analogously, the comultiplication on $\mathcal{O}(G)$ is equivariant for the representations r_A and $\mathrm{id}_{\mathcal{O}(G)\otimes_k R} \otimes r_A$, and therefore by (1) and (3) the diagram

$$\begin{array}{ccc} \mathcal{O}(G) \otimes_k R & \xrightarrow{\Delta_R} & \mathcal{O}(G) \otimes_k R \otimes_k \mathcal{O}(G) \otimes_k R \\ \lambda_A \downarrow & & \downarrow \mathrm{id}_{\mathcal{O}(G)} \otimes \lambda_A \\ \mathcal{O}(G) \otimes_k R & \xrightarrow[\Delta_R]{} & \mathcal{O}(G) \otimes_k R \otimes_k \mathcal{O}(G) \otimes_k R \end{array}$$

commutes. Thus, using the previous lemma on the affine scheme G_R over R, we get $g \in G(R)$ such that $\lambda_A = (r_A)_R(g)$.

Now let (V, r_V) be some finitely generated representation of G. By proposition 3.19, $\rho : V \to V \otimes_k \mathcal{O}(G)$ is an injective homomorphism of representations, where the representation on $V \otimes_k \mathcal{O}(G)$ is taken to be the tensor product of trivial and regular representation. I just showed that $(r_{(-)})_R(g)$ and $\lambda_{(-)}$ agree on the regular representation, and by (2) they agree on the trivial representation on V. Therefore, they also agree on the tensor product, and then by injectivity of ρ and (3), we finally get $\lambda_V = (r_V)_R(g)$. This proves existence. Uniqueness follows trivially from the fact that the regular representation is faithful. □

Since every element $g \in G(R)$ of an affine monoid defines such a family of morphisms by $\lambda_V = (r_V)_R(g)$, the theorem shows that the category of finitely generated linear representations of an affine monoid G encompasses all the information about the monoid itself, or in other words, we can recover G from the category $\mathbf{Rep}(G)$. This is also the reason this theorem is called *reconstruction theorem.*

Now the question arises how we can implement this reconstruction when given a category which we know to be of the form $\mathbf{Rep}(G)$. To this end, note that every such category comes with a family of forgetful functors

$$\omega_R : \mathbf{Rep}(G) \to R\mathbf{Mod}, \ (V, r_V) \rightsquigarrow V \otimes_k R$$

indexed by k-algebras R. Let $\mathrm{End}^{\otimes}(\omega_R)$ be the monoid of natural endomorphisms of ω_R respecting the monoidal structure. These requirements are the same stating that every $\lambda \in \mathrm{End}^{\otimes}(\omega_R)$ satisfies conditions (1) and (2) from theorem 4.2, and condition (3) is equivalent to naturality. We thus get the following

Corollary 4.4 (Tannaka duality for affine monoids)**.** *Let G be an affine monoid. For every k-algebra R, the monoid morphism*

$$G(R) \to End^{\otimes}(\omega_R)$$

which sends $g \in G(R)$ to $(r_{(-)})_R(g)$ is an isomorphism, and the correspondence is natural in R. In particular, if $\underline{End}^{\otimes}(\omega)$ denotes the functor $R \rightsquigarrow End^{\otimes}(\omega_R)$, we have

$$G \simeq \underline{End}^{\otimes}(\omega).$$

4.3 The recognition theorem

In this section, k is always assumed to be an algebraically closed field of characteristic zero. In particular, all k-modules are free and $\underline{\mathrm{End}}(V)$ always is an affine monoid. As discussed before, the category of finite dimensional comodules over some cogebra $\mathcal{C}$ is abelian k-linear and comes with an exact and faithful fiber functor to $\mathbf{Vect}_k$. Further, this category is clearly essentially small, meaning equivalent to a small category, because $\mathbf{Vect}_k$ is. I will now prove that every category with these characteristics is already a representation category; this is called the *recognition theorem.*

Definition 4.5. For an abelian category $\mathbf{C}$ and an object X, denote by $[X]$ the isomorphism class of X in $\mathbf{C}$. Also, denote by $\langle X\rangle$ the full subcategory of $\mathbf{C}$ whose objects are subquotients of direct sums of copies of X. We have a partial order on the isomorphism classes given by $[X] \leqslant [Y]$ if $\langle X\rangle \subset \langle Y\rangle$, and since $[X], [Y] \leqslant [X \oplus Y]$, for essentially small categories this partial order is a projective system.

Theorem 4.6 (Recognition theorem for comodules)**.** *Let $\boldsymbol{C}$ be an essentially small k-linear abelian category, and let*

$$\omega : \boldsymbol{C} \to \boldsymbol{Vect}_k$$

be a k-linear exact and faithful functor. Then there is a cogebra $\mathcal{C}$ such that we have an equivalence

$$\boldsymbol{C} \simeq \boldsymbol{Comod}\mathcal{C}.$$

In particular, if $\mathcal{C}(\omega)$ is the cogebra $\varinjlim_{[X]} \operatorname{End}(\omega|_{\langle X\rangle})^\vee$, then for every object X the vector space $\omega(X)$ has the structure of a right $\mathcal{C}(\omega)$-comodule, and ω defines an equivalence between $\boldsymbol{C}$ and $\boldsymbol{Comod}\mathcal{C}(\omega)$.

For the proof of this theorem we first need several preliminary results. Let $\mathbf{C}$ and ω be as in the theorem.

Lemma 4.7. *1. For all objects X and Y in $\boldsymbol{C}$, $\operatorname{Hom}_{\boldsymbol{C}}(X, Y)$ is finite dimensional over k.*

2. The fiber functor ω reflects monomorphisms, epimorphisms and isomorphisms.

Proof. The first statement is a direct consequence of the fact that ω is faithful. The second statement is just a standard result on faithful functors (see e.g. [ML78]). □

We will first work in the setting of the category $\langle X\rangle$. For a set $S \subset \omega(X)$, the intersection[6] of all subobjects Y of X such that $S \subset \omega(Y)$ is called the *subobject generated by S*. In particular,

Definition 4.8. An object Y is called *monogenic* if it is generated by a single element, i.e. there is a *generator* $y \in \omega(Y)$ such that for each $Y' \subset Y$, $y \in \omega(Y')$ implies $Y' = Y$.

Lemma 4.9. *For every monogenic object Y in $\langle X\rangle$, we have*

$$\dim \omega(Y) \leqslant (\dim \omega(X))^2.$$

Proof. Let Y be a subquotient, i.e. we have morphisms $Y \twoheadleftarrow Y_1 \hookrightarrow X^m$. If $y_1 \in \omega(Y_1)$ is a preimage of $y \in \omega(Y)$ and Z is the subobject of Y_1 generated by y_1, then the image of Z in Y contains y, and is therefore equal to Y. Thus, WOLOG assume that $Y \subset X^m$ for some $m \in \mathbb{N}$, and suppose that $m > \dim \omega(X)$. Since the generator y of Y lies in $\omega(Y) \subset \omega(X)^m$, we can write $y = (y_1, \ldots, y_m)$ with $y_i \in \omega(X)$ linearly dependent. Thus we can find $\lambda_i \in k$ not all zero such that $\sum_i \lambda_i y_i = 0$. These λ_i define a morphism $\lambda : X^m \to X$, which is surjective when pushed down to $\mathbf{Vect}_k$. Therefore we can find a matrix A that extends λ to an isomorphism,

$$\begin{pmatrix} \lambda_1 \ldots \lambda_m \\ A \end{pmatrix} : X^m \to X^m.$$

Let N be the kernel of λ. Then $\omega(A)$ sends $\omega(N)$ to $\omega(X)^{m-1}$, and this is an isomorphism. Since ω reflects isomorphisms, this shows that $N \simeq X^{m-1}$, and $y \in \omega(N)$ implies that Y embeds to X^{m-1}. We can continue in this fashion until we have $m' \in \mathbb{N}$ with $Y \subset X^m$ and $m \leqslant \dim \omega(X)$. This proves the claim. □

[6] The intersection of two subobjects is just their categorical product in the poset category of subobjects.

Lemma 4.10. *Let P be a monogenic object in $\langle X\rangle$ with largest possible* $\dim\omega(P)$*, and let p be a generator of P. Then (P,p) represents $\omega|_{\langle X\rangle}$.*

Proof. Since $\omega(X)$ is of finite dimension, the previous lemma shows that $\dim\omega(Y)$ can take on only finitely many values for monogenic Y, so such a P exists. Since every object in $\langle X\rangle$ can be written as a subquotient of some direct power of X, it now suffices to show that for every $x \in \omega(X)$ there exists a unique morphism $f : P \to X$ such that $\omega(f)(p) = x$.
Uniqueness: Suppose f and g are two such morphisms. Then look at the equalizer

$$E \longrightarrow P \overset{f}{\underset{g}{\rightrightarrows}} X$$

Since E is a subobject of P and by construction $p \in \omega(E)$, we must have $E = P$ and therefore $f = g$.
Existence: Let Q be the smallest subobject of the product $P \amalg X$ such that $\omega(Q)$ contains (p,x). Since p generates P, the image of the projection onto the first factor is all of P, which implies $\dim\omega(Q) \geqslant \dim\omega(P)$. But we chose P maximal monogenic, and thus this projection actually is an isomorphism. The image of its inverse under ω sends p to (p,x), and composed with the image of the projection to the second factor we get the desired morphism. □

In particular, this shows that $\mathrm{Hom}_{\mathbf{C}}(P,-)$ is exact and faithful. In other words, P is a projective generator of $\mathbf{C}$. We are now in the position the make the following definition.

Definition 4.11. Let $A := \mathrm{End}_{\mathbf{C}}(P) = \mathrm{End}(\omega|_{\langle X\rangle})$ as a k-algebra. For all objects Y in $\langle X\rangle$, $\mathrm{Hom}_{\mathbf{C}}(P,Y)$ is a right A-module in the canonical way. Since P represents ω, $\omega(Y)$ has the structure of a left A-module, and we can thus regard ω (resp. $\mathrm{Hom}_{\mathbf{C}}(P,-)$) as a functor from $\langle X\rangle$ to left (resp. right) A-modules.

Lemma 4.12. *The functor $\omega|_{\langle X\rangle}$ is an equivalence of categories $\langle X\rangle \to A\mathbf{Mod}$.*

Proof. The functor $\omega|_{\langle X\rangle}$ is faithful by definition, so we need to show that it is full and essentially surjective.
Essentially surjective: For an A-module M, choose a finite presentation

$$A^m \xrightarrow{u} A^n \to M \to 0.$$

Then u can be viewed as a $m \times n$-Matrix with entries in A, and in this manner it also defines a morphism $P^m \to P^n$. If Y is its cokernel, the exactness of $\mathrm{Hom}_{\mathbf{C}}(P,-)$ implies that $M \simeq \mathrm{Hom}_{\mathbf{C}}(P,Y) \simeq \omega(Y)$.
Full: By the above every object Y of $\langle X\rangle$ occurs in an exact sequence

$$P^m \xrightarrow{u} P^n \to Y \to 0.$$

Let Y' be another object. Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{C}}(P^m,Y') = \mathrm{Hom}_{\mathbf{C}}(P,Y')^m &\simeq \mathrm{Hom}_{A\mathbf{Mod}}(A^m,\mathrm{Hom}_{\mathbf{C}}(P,Y'))\\ &= \mathrm{Hom}_{A\mathbf{Mod}}(\mathrm{Hom}_{\mathbf{C}}(P,P^m),\mathrm{Hom}_{\mathbf{C}}(P,Y')),\end{aligned}$$

where the second equality follows from the fact that A is the tensor unit of $A\mathbf{Mod}$. Now the commuting diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(Y,Y') & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(P^n,Y') & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(P^m,Y') \\ & \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ 0 \longrightarrow & \mathrm{Hom}_{A\mathbf{Mod}}(\omega(Y),\omega(Y')) & \longrightarrow & \mathrm{Hom}_{A\mathbf{Mod}}(A^n,\omega(Y')) & \longrightarrow & \mathrm{Hom}_{A\mathbf{Mod}}(A^m,\omega(Y')) \end{array}$$

shows that we have $\mathrm{Hom}_{\mathbf{C}}(Y, Y') \simeq \mathrm{Hom}_{A\mathbf{Mod}}(\omega(Y), \omega(Y'))$, and this isomorphism is given by $\mathrm{Hom}_{\mathbf{C}}(P, -)$. □

We can now finally formulate a proof for the recognition theorem for comodules.

Proof of theorem 4.6. For any object X of $\mathbf{C}$, let $A_X := \mathrm{End}(\omega|_{\langle X\rangle})$ and let $\mathcal{C}_X := A_X^\vee$, which is a cogebra since A_X is finite dimensional for all X. Furthermore, a left A_X-module yields a right $\mathcal{C}_X$-comodule, and lemma 4.12 shows that the functor $\omega|_{\langle X\rangle}$ is an equivalence of categories between $\langle X\rangle$ and $\mathbf{Comod}\mathcal{C}_X$. Now if $[X] \leqslant [Y]$ for two objects X and Y in $\mathbf{C}$, we have a restriction morphism $A_Y \to A_X$ whose dual yields a morphism $\mathcal{C}_X \to \mathcal{C}_Y$. We can then take the directed limit

$$\mathcal{C}(\omega) := \varinjlim_{[X]} \mathrm{End}(\omega|_{\langle X\rangle})^\vee$$

over these morphisms. Then every $\omega(X)$ for some object X has the structure of a right $\mathcal{C}(\omega)$-comodule, and ω is an equivalence of categories between $\mathbf{C}$ and $\mathbf{Comod}\mathcal{C}(\omega)$ by construction. This completes the proof. □

On the correspondence between bimodule structures and tensor products. Since we are not primarily interested in the category of comodules over some cogebra, but rather in the category of linear representations of some affine group, it remains to find out how this recognition theorem interacts with the extra structure of the coordinate ring of an affine group in comparison with a general cogebra. In section 3.5 I showed how a bialgebra structure on a cogebra yields a monoidal structure on its category of representations. Actually, the converse is also true, as I will now prove.

For some finite-dimensional k-algebra A and an arbitrary k-algebra R, let $\omega_R : A\mathbf{Mod} \to R\mathbf{Mod}$ be the functor sending an A-module M to the free R-module $R \otimes_k M$.

Lemma 4.13. *The canonical map*

$$u : R \otimes_k A \to End(\omega_R), \quad u(r \otimes a)_M(s \otimes m) = rs \otimes am$$

*is an isomorphism. In particular, $A \simeq End(\omega)$, where ω is just the forgetful functor on A**Mod**.*

Proof. I will show that the map $v : \mathrm{End}(\omega_R) \to R \otimes_k A$ sending a natural transformation λ to $\lambda_A(1 \otimes 1)$ is an inverse to u. Clearly, $v \circ u = \mathrm{id}_{R\otimes_k A}$, so we only have to show $u \circ v = \mathrm{id}_{\mathrm{End}(\omega_R)}$. For any $\lambda \in \mathrm{End}(\omega_R)$ we have $\lambda_{A\otimes_k M} = \lambda_A \otimes \mathrm{id}_M$ because $A \otimes_k M$ is a direct sum of copies of A. Let $\mu : A \otimes_k M \to M$ be the multiplication map $a \otimes m \mapsto am$. Then μ is A-linear, and by naturality of λ the diagram

$$\begin{array}{ccc} R \otimes_k A \otimes_k M & \xrightarrow{\mathrm{id}_R \otimes \mu} & R \otimes_k M \\ \downarrow{\scriptstyle \lambda_A \otimes \mathrm{id}_M} & & \downarrow{\scriptstyle \lambda_M} \\ R \otimes_k A \otimes_k M & \xrightarrow[\mathrm{id}_R \otimes \mu]{} & R \otimes_k M \end{array}$$

commutes. But starting with $1 \otimes 1 \otimes m \in R \otimes_k A \otimes_k M$, this means

$$\lambda_M(1 \otimes m) = \lambda_M(1 \otimes \mu(1 \otimes m)) = (\mathrm{id}_R \otimes \mu)(\lambda_A(1 \otimes 1) \otimes m) = u(\lambda_A(1 \otimes 1))_M(1 \otimes m),$$

so the claim holds. □

We can also use this result on cogebras. If $\mathcal{C}$ is some cogebra, then $\mathcal{C}^\vee$ is an algebra, and since every comodule is the projective limit of all its finite-dimensional subcomodules, we then get

$$\mathcal{C} \simeq \varinjlim_{[M]} \mathrm{End}(\omega|_{\langle M\rangle})^\vee.$$

With this in mind, we can prove

Proposition 4.14. *There is a one-to-one correspondence between cogebra homomorphisms $u : \mathcal{C} \to \mathcal{C}'$ and functors $F : \mathbf{Comod}\mathcal{C} \to \mathbf{Comod}\mathcal{C}'$ such that $\omega_{\mathcal{C}'} \circ F = \omega_{\mathcal{C}}$. This correspondence is given by sending u to the functor that sends a $\mathcal{C}$-comodule (V,ρ) to the $\mathcal{C}'$-comodule $(V, (id_V \otimes u)\circ\rho)$.*

Proof. A natural transformation $\lambda \in \mathrm{End}(\omega_{\mathcal{C}'}|_{\langle Y\rangle})$ is already determined by λ_Y, since λ is additive and naturality then gives the right morphisms for quotient- and subobjects of Y^m; and the same holds for $\mathcal{C}$. Therefore F determines a morphism $\tilde{F} : \mathrm{End}(\omega_{\mathcal{C}'}|_{\langle FX\rangle}) \to \mathrm{End}(\omega_{\mathcal{C}}|_{\langle X\rangle})$ by $\tilde{F}(\lambda)_X := \lambda_{FX}$, which then also gives a morphism

$$\varinjlim_{[X]} \mathrm{End}(\omega_{\mathcal{C}'}|_{\langle FX\rangle}) \to \varinjlim_{[X]} \mathrm{End}(\omega_{\mathcal{C}}|_{\langle X\rangle}).$$

Since the LHS is a quotient of $\varinjlim_{[Y]} \mathrm{End}(\omega_{\mathcal{C}'}|_{\langle Y\rangle})$, precomposing with the projection and taking duals yields the desired morphism $\mathcal{C} \to \mathcal{C}'$. □

As we want to use this knowledge to look at the correspondence between monoidal structures on **Comod**$\mathcal{C}$ and bialgbra structures on $\mathcal{C}$, let us apply this proposition to a cogebra $\mathcal{C}$ and its tensor square $\mathcal{C} \otimes_k \mathcal{C}$, which has canonical cogebra structure $(\Delta \otimes \Delta, \epsilon \otimes \epsilon)$. Define

$$\omega \otimes \omega : \mathbf{Comod}\mathcal{C} \times \mathbf{Comod}\mathcal{C} \to \mathbf{Vect}_k$$

by $(\omega \otimes \omega)((V,\rho),(W,\delta)) = V \otimes_k W$. Then using the same argumentation as in lemma 4.13, one can see that for finite dimensional cogebras $\mathcal{C}$ one has $\mathrm{End}(\omega\otimes\omega) \simeq \mathcal{C}\otimes_k\mathcal{C}$, and therefore in general

$$(\mathbf{Comod}\mathcal{C} \times \mathbf{Comod}\mathcal{C}, \omega_{\mathcal{C}} \otimes \omega_{\mathcal{C}}) \simeq (\mathbf{Comod}(\mathcal{C} \otimes_k \mathcal{C}), \omega_{\mathcal{C}\otimes_k\mathcal{C}}).$$

As a result, we finally get the following

Proposition 4.15. *There is a one-to-one correspondence between cogebra homomorphisms $m : \mathcal{C} \otimes_k \mathcal{C} \to \mathcal{C}$ and bilinear functors ϕ : **Comod**$\mathcal{C}$ × **Comod**$\mathcal{C}$ → **Comod**$\mathcal{C}$ satisfying $\phi(V,W) = V \otimes_k W$ as vector spaces.*

In particular, this correspondence is given by $m \mapsto \phi^m$ with ϕ^m sending $(V,\rho),(W,\delta)$ to $V \otimes_k W$ with coaction

$$V \otimes_k W \xrightarrow{\rho\otimes\delta} V \otimes_k \mathcal{C} \otimes_k W \otimes_k \mathcal{C} \xrightarrow{\simeq} V \otimes_k W \otimes_k \mathcal{C} \otimes_k \mathcal{C} \xrightarrow{id_V \otimes id_W \otimes m} V \otimes_k W \otimes \mathcal{C}.$$

Proof. This is a direct consequence of the above discussion and lemma 4.13. The second part follows since $(V,\rho),(W,\delta) \mapsto (V \otimes_k W, \rho\otimes\delta)$ is an isomorphism **Comod**$\mathcal{C}$ × **Comod**$\mathcal{C} \simeq$ **Comod**$(\mathcal{C} \otimes_k \mathcal{C})$. □

The additional structure of a category of representations of an affine monoid in comparison to a category of comodules is precisely the existence on a monoidal structure.

Corollary 4.16 (Recognition theorem for affine monoids)**.** *Let $\boldsymbol{C}$ be an essentially small k-linear abelian category with an exact and faithful fiber functor $\omega : \boldsymbol{C} \to \boldsymbol{Vect}_k$, and let $\otimes : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ be a k-bilinear functor*[7] *such that*

[7] I will write $\otimes(X,Y)$ as $X \otimes Y$.

(1) $\omega(X \otimes Y) = \omega(X) \otimes_k \omega(Y)$*;*

(2) There are natural isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ *and* $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ *whose images under* ω *are the usual associativity and commutativity isomorphisms of the tensor product of vector spaces;*

(3) There is an object $I \in \boldsymbol{C}$ *such that* $\omega(I) = k$ *and we have isomorphisms* $X \otimes I \simeq X \simeq I \otimes X$ *whose images are the canonical isomorphisms making* k *the tensor unit of vector spaces.*

Then the cogebra $\mathcal{C}(\omega)$ *from theorem* 4.6 *has a unique bialgebra structure* (m, e) *such that* $\otimes = \phi^m$ *and the comodule structure on* $\omega(I)$ *is given by* $k \xrightarrow{e} \mathcal{C}(\omega) \simeq k \otimes_k \mathcal{C}(\omega)$*.*

Proof. The conditions (1)-(3) make sure that $\omega : \mathbf{C} \to \mathbf{Comod}\mathcal{C}(\omega)$ preserves the structure associated with $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Therefore, this follows directly from proposition 4.15. □

This is exactly the extension of the recognition theorem 4.6 to the case of affine monoids, since bialgebras are the monoid objects in the category $\mathbf{Alg}_k^{\mathrm{op}}$. In the last step we want to further extend the theorem to affine groups. To do this, first note the following.

Proposition 4.17. *Let* $\boldsymbol{C}$ *and* $\boldsymbol{C'}$ *be symmetric monoidal categories and let* $F, G : \boldsymbol{C} \to \boldsymbol{C'}$ *be strong monoidal functors. If* $\boldsymbol{C}$ *and* $\boldsymbol{C'}$ *are rigid, then every morphism* $\lambda : F \to G$ *of monoidal functors is an isomorphism.*

Proof. The morphism $\mu : G \to F$ making the diagram

$$\begin{array}{ccc} F(X^\vee) & \xrightarrow{\lambda_{X^\vee}} & G(X^\vee) \\ \simeq\downarrow & & \downarrow\simeq \\ F(X)^\vee & \xrightarrow[(\mu_X)^\vee]{} & G(X)^\vee \end{array}$$

commute for all objects X in $\mathbf{C}$ is an inverse for λ. □

Finally, we can state the missing condition for the affine monoid in corollary 4.16 to be an affine group.

Corollary 4.18. *Let* $\boldsymbol{C}$ *and* ω *be as in corollary* 4.16*. Then the cogebra* $\mathcal{C}(\omega)$ *belongs to an affine monoid* G*, and if* $\boldsymbol{C}$ *is rigid then* G *is in fact an affine group.*

Proof. Condition (1) of corollary 4.16 implies that the tensor product on $\mathbf{C}$ is symmetric, so we can use the previous proposition on $\omega : \mathbf{C} \to \mathbf{Rep}(G)$. Since by the reconstruction theorem we have $G \simeq \underline{\mathrm{End}}^{\otimes}(\omega)$, this shows that every element of $G(R)$ is invertible for all k-algebras R, and therefore G is a group. □

While this is a very nice result, the statements of corollary 4.16 and 4.18 are still somewhat disorganized and difficult to grasp. To improve on that, note that the conditions in corollary 4.16 can be summarized in terms of monoidal categories, since they simply state that the category in question is symmetric monoidal and that ω is functor of monoidal categories. In particular, we make the following definition.

Definition 4.19 (Neutral Tannakian category)**.** A non-trivial essentially small abelian k-linear category $\mathbf{C}$ is called *neutral Tannakian* if it is symmetric monoidal with k-bilinear tensor product, rigid, and equipped with a strong monoidal k-linear exact and faithful functor $\omega : \mathbf{C} \to \mathbf{Vect}_k$. Such a functor is called *fiber functor.*

Remark 4.20. The condition that **C** be non-trivial is most often stated as $\mathrm{End}_{\mathbf{C}}(I) = k$ for I the tensor unit. Indeed, if **C** satisfies all the conditions of definition, we have $\omega(I) = k$ and since ω is faithful k-linear, $\mathrm{End}_{\mathbf{C}}(I) \subset \mathrm{End}_{\mathbf{Vect}_k}(\omega(I)) = \mathrm{End}_{\mathbf{Vect}_k}(k) = k$ is a linear subspace, and then non-triviality of **C** implies that it must be a non-trivial subspace. The necessity of this condition stems from the fact that no group has empty representation theory; there always is a trivial action.

Definition 4.19 enables us to state the results of this section in a more coherent manner. As a final result, we get

Theorem 4.21 (Recognition theorem for affine groups)**.** *Every neutral Tannakian category is equivalent as symmetric monoidal category to the category of representations of some affine group. In particular, if ω is a fiber functor for the category, the corresponding group is given by*

$$G \simeq \underline{\mathrm{End}}^{\otimes}(\omega).$$

5 Tannaka duality for affine supergroups

The previous section leaves the question as to when exactly a k-linear monoidal category admits a fiber functor. It turns out that to answer this question in the most general context, we need to allow a fiber functor not to the category of finite dimensional vector spaces, but to the category of finite dimensional *super vector spaces*, of which the category of usual vector spaces is a full subcategory. This will lead to the generalization of affine group schemes to affine supergroups.

Throughout this section, let k be an algebraically closed field of characteristics zero.

5.1 Super linear algebra and supergroups

We first of all need to investigate the category of super vector spaces and in particular the theory of algebras in this theory. We start from the very general definition of a *graded object*.

Definition 5.1. Let S be a set and let **C** be some category. Then the category of S-graded objects of **C** is defined to be the functor category $\mathbf{C}^S$ of functors from S to **C**, where S is viewed as discrete category. In particular, a S-graded object of **C** is a map X assigning an object X_s of **C** to each element $s \in S$, and a morphism f of S-graded objects X and Y is a family of morphisms $f_s : X_s \to Y_s$.

If **C** is rigid, k-linear abelian and symmetric monoidal with tensor unit I, this structure can be transported to the category of graded objects of **C**.

Proposition 5.2. *If S is a monoid with neutral element e, then we have a tensor product in $\mathbf{C}^S$ given by*

$$(X \otimes Y)_s = \bigoplus_{ab=s} X_a \otimes Y_b$$

with unit $I^S_e = I$ and $I^S_s = 0$ if $s \neq e$ making $\mathbf{C}^S$ a monoidal category. If S is a commutative monoid, this monoidal category is symmetric. If S is a group, then every object in $\mathbf{C}^S$ has a left and a right dual, and if it even is an abelian group, then those agree.

Proof. Associativity and unit constraint are directly inherited from **C**, as well as the corresponding pentagon and triangle identity. If S is commutative and τ is a braiding in **C**, then $ab = ba$ for all $a, b \in S$ and

$$(X \otimes Y)_s = \bigoplus_{ab=s} X_a \otimes Y_b \xrightarrow{\bigoplus \tau_{X_a,Y_b}} \bigoplus_{ba=s} Y_b \otimes X_a = (Y \otimes X)_s$$

is a braiding in $\mathbf{C}^S$. If S has inverses we can built a right dual object to X by $(X^\vee)_s = (X^\vee_{s^{-1}})$ (analogously for left dual), and a given braiding in $\mathbf{C}^S$ makes sure that left and right duals agree. □

If $\mathbf{C}$ is abelian, $\mathbf{C}^S$ inherits the abelian structure by setting

$$(X \oplus Y)_s = X_s \oplus Y_s.$$

We are particularly interested in the case $\mathbf{C} = \mathbf{Vect}_k$ and $S = \mathbb{Z}_2$.

Definition 5.3 (Super vector space, part 1)**.** A $\mathbb{Z}_2$-graded object of $\mathbf{Vect}_k$ is called a *super vector space*. In particular, a super vector space V is given by two finite dimensional vector spaces V_0 and V_1[8], which are called the *even* and *odd* component of V respectively. A morphisms of super vector spaces V and W is simply given by two linear maps $f_0 : V_0 \to W_0$ and $f_1 : V_1 \to W_1$.

The category of super vector spaces is clearly equivalent to the category whose objects are finite dimensional vector spaces V together with a direct sum decomposition $V = V_0 \oplus V_1$ into two factors, and whose morphisms are linear maps preserving these decompositions. This is the version more often encountered in the literature, and I will use this notation from now on. In this point of view, an non-zero element v of a super vector space V is called *homogeneous* if it lies in either V_0 or V_1, and for a homogeneous v define $|v| = 0$ if $v \in V_0$ and $|v| = 1$ if $v \in V_1$. It always suffices to define morphisms of super vector spaces on homogeneous elements only.

Since $\mathbb{Z}_2$ is an abelian group, we have a natural tensor product making $\mathbf{Vect}_k^{\mathbb{Z}_2}$ into a monoidal category, i.e. for super vector spaces $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$,

$$V \otimes W = (V_0 \otimes W_0 \oplus V_1 \otimes W_1) \oplus (V_0 \otimes W_1 \oplus V_1 \otimes W_0)$$

with the brackets indicating the new decomposition. The unit object is the field k viewed as purely even super vector space, which I will denote by $k^{1|0}$. However, we choose a different braiding than the one inherited from the trivial braiding on $\mathbf{Vect}_k$. In particular, we choose

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \tag{4}$$

for homogeneous $v \in V$ and $w \in W$. This braiding is often called *Koszul rule*. The fact that we choose a non-trivial braiding here, thus making the category of supervector spaces into a different symmetric monoidal category, is the very foundation of super geometry in mathematics and supersymmetry in physics.

Definition 5.4 (Super vector space, part 2)**.** Denote by $\mathbf{sVect}_k$ the k-linear abelian symmetric monoidal category of super vector spaces over the field k. The dimension of a super vector space $V = V_0 \oplus V_1$ is defined to be $\dim V_0 | \dim V_1$.

Of course the concept of a super vector space (and associated concepts) can be generalized to the category of possibly infinite dimensional vector spaces. We have an important automorphism of every super vector space $V = V_0 \oplus V_1$ given

$$\Pi : V \to V, \quad v \mapsto (-1)^{|v|} v$$

for homogeneous $v \in V$. It is called the *parity automorphism*.

Just like in the non-graded case, the next logical step is to look at structures in the category of super vector spaces. An ordinary k-algebra is a commutative monoid in the category of vector spaces, and analogously we define

[8] I will write 0 and 1 here instead of the more appropriate $\bar{0}$ and $\bar{1}$.

Definition 5.5. A (finite dimensional) *superalgebra* over k is a commutative monoid in the category $\mathbf{sVect}_k$. In particular it is given by a super vector space A together with a multiplication

$$m : A \otimes A \to A, \quad a \otimes b \mapsto a \cdot b$$

and a unit $e : k^{1|0} \to A$ which are both morphisms of super vector spaces and satisfy the usual identities. By the non-trivial braiding we always get

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

for homogeneous $a, b \in A$. A morphism of superalgebras is a morphism of super vector spaces $f : A \to B$ such that $m_B \circ f \otimes f = f \circ m_A$ and $e_B = f \circ e_A$.

Analogously, we can define a super cogebra to be a cocommutative comonoid in $\mathbf{sVect}_k$. A super bialgebra then is a super vector space which is simultaneously a super algebra and a super cogebra such that the comultiplication and counit are morphisms of super algebras. If we have a super bialgebra H, an antipode $S : H \to H$ which is a morphism of super bialgebras makes H into a *super Hopf algebra* if the usual antipode diagram commutes (see section 2.4).

Example 5.6. Let $F(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ be the algebra of non-commuting polynomials in the variables X_i and Y_j, i.e. the free k-algebra over these symbols. Then the quotient of this algebra by the relations

$$X_i X_{i'} = X_{i'} X_i, \quad X_i Y_j = Y_j X_i, \quad Y_j Y_{j'} = -Y_{j'} Y_j, \quad i = 1, \ldots, n;\ j = 1, \ldots, m$$

is a superalgebra over k; the *super polynomial algebra* in even variables X_i, $i = 1, \ldots, n$ and odd variables Y_j, $j = 1, \ldots, m$. In particular, if this algebra is denoted by $\mathrm{s}k[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$, we have the decomposition

$$\mathrm{s}k[X_1, \ldots, X_n; Y_1, \ldots, Y_m]_0 = \left\{ f_0 + \sum_I f_I Y_I : I = \{i_1 < \cdots < i_r\}, r \,\text{even} \right\}$$
$$\mathrm{s}k[X_1, \ldots, X_n; Y_1, \ldots, Y_m]_1 = \left\{ \sum_J f_J Y_J : J = \{j_1 < \cdots < j_s\}, s \,\text{odd} \right\},$$

with I and J understood as multi-indices and $f_0, f_{i_k}, f_{j_l} \in k[X_1, \ldots, X_n]$.

Definition 5.7. Let A be a superalgebra over k. A left A-module is a super vector space M together with a morphism $\rho : A \otimes M \to M$ of super vector spaces satisfying the usual identities. Such a left A-module can be viewed as an A-bimodule by defining

$$\rho(m \otimes a) = (-1)^{|m||a|} \rho(a \otimes m)$$

for $a \in A$ and $m \in M$. A morphism of A-modules is a morphism of super vector spaces $f : M \to N$ such that $f \circ \rho_M = \rho_N \circ \mathrm{id}_A \otimes f$. As in the non-graded case, the category of modules over a superalgebra is a symmetric monoidal abelian category.

We have seen in section 3.2 that an affine group is given by a functor $G : \mathbf{Alg}_k \to \mathbf{Set}$ represented by some Hopf algebra $\mathcal{O}(G)$. Now that I have defined super Hopf algebras, it is natural to generalize this concept to that of an *affine supergroup*.

Definition 5.8 (Affine supergroup)**.** An affine supergroup is a group object in the category of representable functors from $\mathbf{sAlg}_k$ to $\mathbf{Set}$. Equivalently, such a representable functor is a supergroup if and only if its representing object is a super Hopf algebra.

One of the easiest examples is the super general linear group, which is the automorphism group of a super vector space $k^{n|m} := k^n \oplus k^m$.

Example 5.9. For $n, m \in \mathbb{N}_0$, let $\mathrm{GL}_{n,m}$ be the functor from $\mathbf{sAlg}_k$ to $\mathbf{Set}$

$$R_0 \oplus R_1 \rightsquigarrow \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in \mathrm{GL}_n(R_0), B \in \mathrm{Mat}_{n,m}(R_1), C \in \mathrm{Mat}_{m,n}(R_1), D \in \mathrm{GL}_m(R_0) \right\}.$$

It is represented by the superalgebra obtained by quotienting $\mathrm{s}k[X_{11}, X_{12}, \dots, X_{m+nm+n}, Y, Z]$ in even symbols Y, Z, X_{ij}, $1 \leqslant i, j \leqslant n$ or $n+1 \leqslant i, j \leqslant n+m$ and odd symbols X_{ij} for all other i, j by the relations

$$Y \cdot \det(X_{ij})_{1 \leqslant i,j \leqslant n} = 1$$
$$Z \cdot \det(X_{ij})_{n+1 \leqslant i,j \leqslant n+m} = 1.$$

More general, for a super vector space V, let $\underline{\mathrm{End}}(V)$ be the functor from superalgebras to sets sending a superalgebra R to $\mathrm{End}_{\mathbf{RMod}}(V \otimes R)$. This enables us to define a representation theory for affine supergroups.

Definition 5.10. Let G be an affine supergroup and let $p \in G(k^{1|0})$ be an element of order two such that the inner automorphism of G induces by p is the parity automorphism. A linear p-representation of G on a super vector space V is a natural transformation

$$r : G \to \underline{\mathrm{End}}(V)$$

of monoid valued functors such that $r_k(p) : V \to V$ is the parity automorphism Π of V. Denote the category of linear p-representation of G by $\mathbf{Rep}(G, p)$.

Remark 5.11. One can translate the conditions imposed on p into the language of Hopf algebras. Recall that if $\mathcal{O}(G)$ is the super Hopf algebra representing G, the multiplication in G translates to the multiplication $f \cdot g = [f, g] \circ \Delta$ on $\mathrm{Hom}_{\mathbf{sAlg}_k}(\mathcal{O}(G), R)$. Thus, if $\Delta(h) = h_{(1)} \otimes h_{(2)}$, then $p^2 = e$ if and only if the map $\mathcal{O}(G) \to k$, $h \mapsto p(h_{(1)})p(h_{(2)})$ is the counit ϵ. Furthermore, the inner automorphism $G \to G$, $g \mapsto pgp^{-1}$ is the parity automorphism of G if and only if the map

$$\mathcal{O}(G) \to \mathcal{O}(G), \quad h \mapsto p(h_{(1)})h_{(2)}p(S(h_{(3)}))$$

is the parity automorphism on $\mathcal{O}(G)$, where $h \mapsto h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ is the two times iterated comultiplication and S the antipode.

Remark 5.12. If $\mathcal{O}(G)$ is purely even, that is, G is an ordinary affine group viewed as an affine super group, then p is central and $\mathbf{Rep}(G, p)$ can be identified as a monoidal category with $\mathbf{Rep}(G)$, but with a different braiding: For each representation (V, r) of G as an ordinary group, $r_k(p)$ defines a $\mathbb{Z}_2$-grading on V by its plus and minus one eigenspaces and the new braiding on $\mathbf{Rep}(G)$ is given by the corresponding Koszul rule. In the physics literature, an ordinary group together with a fixed element of the center is often already called a supergroup, because on can then consider "super representation" of this group (see section 6.3).

5.2 Super Tannakian formalism

In the remainder of this section we will always work with certain categories called *tensor categories*. They are designed to model Tannakian categories leaving out the requirement of a fiber functor.

Definition 5.13 (Tensor category)**.** A category $\mathcal{A}$ is called a tensor category over k if it is essentially small, abelian k-linear, symmetric monoidal with the tensor product being bilinear and exact in each variable, rigid, and satisfies and $\mathrm{End}_{\mathcal{A}}(I) = k$. A fiber functor between tensor categories is a faithful k-linear exact strong monoidal functor.

In section 4.3 I showed that having a fiber functor into the category of vector spaces suffices for a tensor category to be of the form $\mathbf{Rep}(G)$ for some affine group G. There is a canonical way to generalize this theorem to fiber functors into the category of super vector spaces, and I will briefly state the corresponding results, which can all be found in section 8 of Deligne's "Catégories tannakiennes" [Del90]. A super fiber functor is a fiber functor into the category of super vector spaces.

Definition 5.14. Let $\mathcal{A}$ be a tensor category and ω a super fiber functor over $\mathcal{A}$. For a superalgebra A, let ω_A be the functor

$$\mathcal{A} \to \mathbf{AMod}, \quad X \rightsquigarrow A \otimes \omega(X).$$

The *automorphism supergroup* of ω is the functor

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) : \mathbf{sAlg}_k \to \mathbf{Set}, \quad A \rightsquigarrow \mathrm{Aut}^{\otimes}(\omega_A).$$

This is indeed an affine supergroup [Del90, 8.11].

Definition 5.15. Let $\mathcal{A}$ be a tensor category and $\mathrm{id}_{\mathcal{A}}$ be its identity functor regarded as a fiber functor from $\mathcal{A}$ to itself. The automorphism supergroup

$$\pi(\mathcal{A}) := \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathcal{A}})$$

is called the *fundamental group* of $\mathcal{A}$.

For the category of super vector spaces we get $\pi(\mathbf{sVect}_k) = \mathbb{Z}_2$ [Del90, 8.14], and the non-trivial element acts as the parity automorphism on super vector spaces. Functors between tensor categories preserve the structure of the fundamental group:

Proposition 5.16. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be tensor categories, and $\eta : \mathcal{A}_1 \to \mathcal{A}_2$ be a k-linear exact strong monoidal functor. Then we have a canonical group homomorphism [Del90, 8.15]*

$$\tilde{\eta} : \pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_1)).$$

The reconstruction theorem for affine supergroups now states [Del90, 8.17 and 8.19]

Theorem 5.17 (Recognition theorem for affine supergroups)**.** *Let $\mathcal{A}$ be a tensor category and $\omega : \mathcal{A} \to \mathbf{sVect}_k$ be super fiber functor. Then we have an equivalence of tensor categories*

$$\mathcal{A} \simeq \boldsymbol{Rep}(\omega(\pi(\mathcal{A})), \tilde{\omega})$$

where $\tilde{\omega}$ is obtained from ω by proposition 5.16.

Remark 5.18. Note that a group homomorphism $\mathbb{Z}_2 \to \omega(\pi(\mathcal{A}_1))$ is equivalent to the choice of an element $p \in \omega(\pi(\mathcal{A}))$ of order two as in definition 5.10, namely the image of the non-trivial element.

Deligne's theorem on tensor categories. In the remainder of this section I will prove Deligne's theorem on tensor categories, which gives a criterion as to when a tensor category admits a super fiber functor, and therefore is of the form $\mathbf{Rep}(G, p)$ for some affine supergroup G.

Definition 5.19 (Subexponential growth)**.** An object X of a tensor category $\mathcal{A}$ is of *subexponential growth* if all its tensor powers are of finite length, and for all $N \in \mathbb{N}$ there is a $n \in \mathbb{N}$ such that

$$\mathrm{length}(X^{\otimes n}) \leqslant N^n.$$

A category $\mathcal{A}$ is of subexponential growth if all its objects are.

Theorem 5.20 (Deligne's theorem on tensor categories)**.** *A k-tensor category is of the form* $\boldsymbol{Rep}(G, p)$ *for some affine supergroup G if and only if it is of subexponential growth.*

In one direction this theorem is trivial, since the length of a super vector space of dimension $p|q$ is $p + q$, and in particular

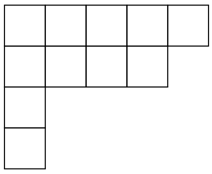
$$\text{length}(X^{\otimes n}) = (p + q)^n.$$

For the other direction I will closely follow Deligne's proof given in "Catégories Tensorielles" [Del02], only filling in details that were left out in the original exposition.

5.3 Some representation theory

I will briefly recall the necessary constructions from classical representation theory needed for the proof of Deligne's theorem.

Representations of the symmetric group. Let n be a natural number. A *partition* of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ of natural numbers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. In this case we call n the order of the partition λ and write $|\lambda| = n$. To a partition λ we can assign a so called *Young diagram* $[\lambda]$ consisting of λ_i boxes in the ith row which are aligned on the left. For example, if $n = 11$ and $\lambda = (5, 4, 1, 1)$, we have the following Young diagram:



I will sometimes use integer coordinates (a, b) to describe the box at the given position in a Young diagram; starting with $(1, 1)$ at the top left corner. The *conjugate partition* λ^t of a partition λ is defined by interchanging rows and columns in the Young diagram of λ, i.e. the Young diagram of λ^t is the one of λ reflected at the diagonal. For the example above, $\lambda^t = (4, 2, 2, 2, 1)$. For two partitions $\mu = (\mu_1, \ldots, \mu_r)$ and $\lambda = (\lambda_1, \ldots, \lambda_s)$ we say that $[\mu] \subset [\lambda]$ if $r \leqslant s$ and $\mu_i \leqslant \lambda_i$ for all $i \leqslant r$. A *Young tableau* is obtained from a Young diagram by numbering the boxes of the diagram. In our example, the canonical Young tableau is

1	2	3	4	5
6	7	8	9	
10				
11				

We have an action of the symmetric group $\mathfrak{S}_n$ on a Young tableau of order n by exchanging the boxes according to the permutation of their numbers. For a given partition λ of n and a numbering of $[\lambda]$, we define two subgroups of $\mathfrak{S}_n$ by

$$P_\lambda := \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each row of } [\lambda]\},$$
$$Q_\lambda := \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each column of } [\lambda]\}$$

The *Young symmetrizer* p_λ is defined to be the product of the two projection operators belonging to P_λ and Q_λ. In particular, if $k[\mathfrak{S}_n]$ denotes the group algebra of the symmetric group,

$$p_\lambda := \left(\sum_{\sigma \in P_\lambda} \sigma\right) \cdot \left(\sum_{\sigma \in Q_\lambda} \text{sgn}(\sigma)\, \sigma\right) \in k[\mathfrak{S}_n].$$

Using this construction, we can classify all irreducible representations of the symmetric group. The following two results can be found in [FH04].

Theorem 5.21. *For a partition λ of n, denote by V_λ the left ideal of the group algebra $k[\mathfrak{S}_n]$ generated by the Young symmetrizer p_λ. Then V_λ is an irreducible representation of $\mathfrak{S}_n$ and every irreducible representation of the symmetric group arises this way for a unique partition λ.*

Proposition 5.22. *Let* sgn *be the alternating representation of the symmetric group, i.e.* $\mathrm{sgn} : \mathfrak{S}_n \to k$, $\sigma \mapsto \mathrm{sgn}(\sigma)$. *Then for every partition λ we have*

$$V_{\lambda^t} = V_\lambda \otimes \mathrm{sgn}.$$

Now let $n_1, \dots, n_r$ be natural numbers summing to n. We can embed the product $\prod_{i=1}^r \mathfrak{S}_{n_i}$ into $\mathfrak{S}_n$ by identifying $\{1, \dots, n\}$ with a disjoint union of the $\{1, \dots, n_i\}$[9]. If μ_i is a partition of n_i for all $i = 1, \dots, r$, then the tensor product $\bigotimes_{i=1}^r V_{\mu_i}$ is an irreducible representation of $\prod_{i=1}^r \mathfrak{S}_{n_i}$. If λ is a partition of n, denote by

$$[\lambda : \mu_1, \dots, \mu_r]$$

the multiplicity of V_λ in the representation of $\mathfrak{S}_n$ induced by this tensor product. These multiplicities are given by the so called *Littlewood-Richardson rule*, which says the following (see [FH04]).

Construction 5.23. Let μ, ν be partitions of a and b respectively, and let $a + b = n$. If $\nu = (\nu_1, \dots, \nu_k)$, a ν-extension of $[\mu]$ is obtained in the following way. First, add ν_1 boxes to the Young diagram $[\mu]$ in such a way that there are never two new boxes in the same column. Then, put the integer 1 in each of the new boxes. Now, add ν_2 boxes to this diagram in the same manner, and put the integer 2 in them. Repeat the process with each $i = 1, \dots, k$. Such a ν-extension is called strict if, when the integers in the boxes are listed from right to left and top to bottom, and one considers the first d entries in this list (for any $1 \leqslant d \leqslant \sum_i \nu_k$), each integer between 1 and $k-1$ occurs at least as many times as its successor.
If λ is a partition of n, the Littlewood-Richardson rule states that the multiplicity $[\lambda : \mu, \nu]$ of V_λ in the induces representation of $\mathfrak{S}_n$ on $V_\mu \otimes V_\nu$ is given by the number of strict ν-extensions of $[\mu]$ to $[\lambda]$.

By transitivity, in the general case we get

$$[\lambda : \mu_1, \dots, \mu_r] = \sum_{\nu_1, \dots, \nu_{r-2}} [\lambda : \nu_1, \mu_r][\nu_1 : \nu_2, \mu_{r-1}] \dots [\nu_{r-3} : \nu_{r-2}, \mu_3][\nu_{r-2} : \mu_1, \mu_2]. \tag{5}$$

Lemma 5.24. *(a) For two partitions μ and λ with $|\mu| \leqslant |\lambda|$ the following are equivalent.*

(i) $[\mu] \subset [\lambda]$

(ii) There exists a partition ν of $(|\lambda| - |\mu|)$ such that $[\lambda : \mu, \nu] \neq 0$

(iii) $[\lambda : \mu, \underbrace{(1), \dots, (1)}_{(|\lambda|-|\mu|) \text{ times}}] \neq 0$

(b) Fix a natural number r and a partition λ of n. Then there exist natural numbers $n_1, \dots, n_r$ summing to n such that $[\lambda : (n_1), \dots, (n_r)] \neq 0$ if and only if $[\lambda]$ has at most r rows.

(c) Fix two natural numbers r, s and a partition λ of n. There exist natural numbers $n_1, \dots, n_r$ and $m_1, \dots, m_s$ summing to n such that $[\lambda : (n_1), \dots, (n_r), (m_1)^t, \dots, (m_s)^t] \neq 0$ if and only if $[\lambda]$ has no box at the place $(r+1, s+1)$.

[9] Specifically, there is an obvious bijection from the set of the first n natural numbers to $\{1, \dots, n_1, n_1+1, \dots, n_1+n_2, \dots, \sum_i n_i\}$

Proof. "(a)" As an immediate consequence of the Littlewood-Richardson rule, if $|\lambda| = |\mu| + 1$ then $[\lambda : \mu, (1)]$ is equal to one if $[\mu] \subset [\lambda]$ and zero else. Expanding $[\lambda : \mu, (1), \ldots, (1)]$ by equation (5), we see that

$$[\lambda : \mu, (1), \ldots, (1)] = \sum_{\nu_1, \ldots, \nu_{k-1}} [\lambda : \nu_1, (1)][\nu_1 : \nu_2, (1)] \ldots [\nu_{k-1} : \mu, (1)],$$

where $k = |\lambda| - |\mu|$. Now all the factors on the right hand side define an inclusion relation between the involved diagrams. By transitivity of inclusion of Young diagrams, this shows that (*i*) and (*iii*) are equivalent. The equivalence of (*ii*) can then simply be seen from the fact that the multiplicities $[\lambda : \mu_1, \ldots, \mu_r]$ are invariant under permutations of the μ_i.

"(b)" In an extension of $[\lambda]$ by $[\mu]$, one may add at most one box per column of $[\lambda]$ for every row of $[\mu]$. Therefore, per row of $[\mu]$, one may increase the number of rows of $[\lambda]$ by a maximum of one. By this reasoning, every direct factor of the induced representation of $V_{(n_1)} \otimes \cdots \otimes V_{(n_r)}$ can have at most r rows.

"(c)" This follows analogously to (b) and can be showed by induction over r and s.

□

The Schur functor. For a finite dimensional vector space V, the general linear group $\mathrm{GL}(V)$ acts on the tensor powers of the vector space $V^{\otimes n}$. We also have a canonical (right) action of the symmetric group on these tensor powers by permuting of the factors, i.e.

$$(v_1 \otimes \cdots \otimes v_n).\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for $v_i \in V$ and $\sigma \in \mathfrak{S}_n$. This right action commutes with the left action of $\mathrm{GL}(V)$. We can thus make the following definition.

Definition 5.25 (Schur functor for vector spaces)**.** Let λ be a partition of n, and let p_λ be the Young symmetrizer corresponding to λ. For a vector space V, define the *Schur functor* S_λ by

$$V \rightsquigarrow S_\lambda(V) := \mathrm{im}(p_\lambda : V^{\otimes n} \to V^{\otimes n}).$$

For every vector space V, this is a representation of $\mathrm{GL}(V)$.

Since as $k[\mathfrak{S}_n]$-modules we trivially have $V^{\otimes n} = k[\mathfrak{S}_n] \otimes_{k[\mathfrak{S}_n]} V^{\otimes n}$, we can also write

$$\begin{aligned} S_\lambda(V) = p_\lambda V^{\otimes n} &= p_\lambda \left(k[\mathfrak{S}_n] \otimes_{k[\mathfrak{S}_n]} V^{\otimes n}\right) = p_\lambda k[\mathfrak{S}_n] \otimes_{k[\mathfrak{S}_n]} V^{\otimes n} \\ &= V_\lambda \otimes_{k[\mathfrak{S}_n]} V^{\otimes n} \simeq V_\lambda^\vee \otimes_{k[\mathfrak{S}_n]} V^{\otimes n} \\ &= \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, V^{\otimes n}). \end{aligned}$$

Since the representations V_λ of $\mathfrak{S}_n$ are self-dual [Jam78], a Schur functor of the dual of a vector space is the dual representation of the Schur functor of the original vector space. The following result shows the importance of these Schur functors (see [FH04]).

Theorem 5.26. *For each partition λ and finite dimensional vector space V, the image of the Schur functor $S_\lambda(V)$ is an irreducible representation of the general linear group $GL(V)$. Further, we have*

$$V^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes S_\lambda(V),$$

where the direct sum ranges over all partition of n. This relation is called Schur-Weyl duality.

I will state some results concerning the behavior of the Schur functor under tensor products and direct sums, partly without proof. They can all be found in [FH04].

Proposition 5.27. *Let V and W be finite dimensional vector spaces over k.*

(a) For partitions μ of k and ν of l we have

$$S_\mu(V) \otimes S_\nu(V) = \bigoplus_{|\lambda|=k+l} S_\lambda(V)^{\oplus[\lambda:\mu,\nu]}.$$

(b) For a partition λ of n we have

$$S_\lambda(V \oplus W) = \bigoplus_{|\mu|+|\nu|=n} (S_\mu(V) \otimes S_\nu(W))^{\oplus[\lambda:\mu,\nu]}.$$

(c) For a partition λ of n we have

$$S_\lambda(V \otimes W) = \bigoplus_{|\mu|=|\nu|=n} (S_\mu(V) \otimes S_\nu(W))^{\oplus[V_\lambda:V_\mu \otimes V_\nu]},$$

with $[V_\lambda : V_\mu \otimes V_\nu]$ the multiplicity of V_λ in the tensor product of V_μ and V_ν.

Corollary 5.28. *If $S_\mu(V) = 0$ for some vector space V, then also $S_\lambda(V) = 0$ for all partitions λ with $[\mu] \subset [\lambda]$.*

Proof. By part (a) of lemma 5.24 there is a partition ν of $(|\lambda| - |\mu|)$ such that $[\lambda : \mu, \nu] \neq 0$. But then $S_\lambda(V)^{\oplus[\lambda:\mu,\nu]}$ is a direct factor of $S_\mu(V) \otimes S_\nu(V) = 0$ by the above proposition, and thus $S_\lambda(V)$ must be zero. □

Generalized Schur functors. Let $\mathcal{A}$ be a k-linear abelian symmetric monoidal category with bilinear and biexact tensor product. We can define an operation between vector spaces and objects of this category by a generalization of the tensor-hom adjunction.

Definition 5.29. Let V be a finite dimensional vector space and let X be an object of the category $\mathcal{A}$ as above. Define $V \otimes X$ to be the object of $\mathcal{A}$ satisfying

$$\mathrm{Hom}_{\mathcal{A}}(V \otimes X, Y) = \mathrm{Hom}_{\mathbf{Vect}_k}(V, \mathrm{Hom}_{\mathcal{A}}(X, Y)) \quad \text{for all objects } Y \text{ of } \mathcal{A}.$$

Also define $\mathrm{Hom}(V, X) := V^\vee \otimes X$. If we choose a basis $\{e_i\}_{i\in I}$ of V, then $V \otimes X$ is isomorphic to $\bigoplus_I X$.

There is an action of the symmetric group $\mathfrak{S}_n$ on objects $X^{\otimes n}$ of $\mathcal{A}$ provided by the braiding. Since every finite dimensional vector space over k can be build (up to isomorphism) out of copies of k, there is only one symmetric monoidal linear functor from $\mathbf{Vect}_k$ to $\mathcal{A}$, and it sends k to the tensor unit I of $\mathcal{A}$. The group algebra of the symmetric group is a monoid in $\mathbf{Vect}_k$, and it thus is sent to a monoid in $\mathcal{A}$ (see proposition 2.23), which I will also denote by $k[\mathfrak{S}_n]$. We can now define the following: For a partition λ of n, we have a generalized Schur functor that sends an objects X of $\mathcal{A}$ to

$$S_\lambda(X) = \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, X^{\otimes n}) \simeq V_\lambda \otimes_{k[\mathfrak{S}_n]} X^{\otimes n}.$$

The results of proposition 5.27 and 5.28 also hold mutatis mutandis for this generalized Schur functor. If in particular we take $\mathcal{A} = \mathbf{sVect}_k$, this yields the following.

Proposition 5.30. *If X is a finite dimensional super vector space of super dimension $p|q$, then $S_\lambda(X) \neq 0$ if and only if $[\lambda]$ has no box at position $(p+1, q+1)$.*

Proof. If Y is a purely odd super vector space with underlying vector space $|Y|$, then the underlying vector space of $Y^{\otimes n}$ is $|Y|^{\otimes n}$ and the action of $\sigma \in \mathfrak{S}_n$ on $|Y^{\otimes n}|$ is $\operatorname{sgn}(\sigma)$ times its natural action on $|Y|^{\otimes n}$. Recall from proposition 5.22 that for a partition ν of n we have $V_{\nu^t} = \operatorname{sgn} \otimes V_\nu$, and therefore

$$|S_\nu(Y)| = |V_\nu \otimes_{k[\mathfrak{S}_n]} Y^{\otimes n}| = V_{\nu^t} \otimes_{k[\mathfrak{S}_n]} |Y|^{\otimes n} = S_{\nu^t}(|Y|).$$

Now for a general super vector space $X = X_0 \oplus X_1$, proposition 5.27 yields

$$\begin{aligned} |S_\lambda(X)| = |S_\lambda(X_0 \oplus X_1)| &= \left| \bigoplus_{|\mu|+|\nu|=|\lambda|} (S_\mu(X_0) \otimes S_\nu(X_1))^{\oplus[\lambda:\mu,\nu]} \right| \\ &= \bigoplus_{|\mu|+|\nu|=|\lambda|} (S_\mu(|X_0|) \otimes S_{\nu^t}(|X_1|))^{\oplus[\lambda:\mu,\nu]} \end{aligned}$$

For any finite dimensional vector space V of dimension n we have $S_{(1,\ldots,1)}V = \Lambda^m V = 0$ for all $m > n$, and therefore by corollary 5.28 also $S_\lambda V = 0$ if $[\lambda]$ has more than n rows. Concludingly, for $S_\lambda(X) \neq 0$ it is necessary and sufficient that there are partitions μ and ν with orders summing to the order of λ such that $[\mu]$ has at most p rows, $[\nu]$ has at most q columns, and $[\lambda : \mu, \nu] \neq 0$. Using lemma 5.24(c), this is equivalent to $[\lambda]$ having no box at position $(p+1, q+1)$. ☐

Corollary 5.31. *Let $p, q, r, s \geqslant 0$ and λ, μ, ν be partitions such that $|\lambda| = |\mu| + |\nu|$. If $(p + r + 1, q + s + 1) \in [\lambda]$ and $[\lambda : \mu, \nu] \neq 0$, then $(p + 1, q + 1) \in [\mu]$ or $(r + 1, s + 1) \in [\nu]$.*

Proof. Let X be a super vector space of dimension $p|q$ and Y a super vector space of dimension $r|s$. By proposition 5.30, we must then have $S_\lambda(X \oplus Y) = 0$, and then proposition 5.27 gives

$$0 = S_\lambda(X \oplus Y) = \bigoplus_{|\mu|+|\nu|=|\lambda|} (S_\mu(X) \otimes S_\nu(Y))^{\oplus[\lambda:\mu,\nu]},$$

so we must have $S_\mu(X) = 0$ or $S_\nu(Y) = 0$ for our specific μ and ν. Again applying proposition 5.30 we see that this entails $(p + 1, q + 1) \in [\mu]$ or $(r + 1, s + 1) \in [\nu]$. ☐

Corollary 5.32. *Let $p, q, r, s \geqslant 0$ and λ, μ, ν be partitions of n. If $(pq + rs + 1, ps + qr + 1) \in [\lambda]$ and $[V_\lambda : V_\mu \otimes V_\nu] \neq 0$, then $(p + 1, q + 1) \in [\mu]$ or $(r + 1, s + 1) \in [\nu]$.*

This follows analogously to the previous corollary by using the tensor product instead of the direct sum.

5.4 Commutative algebra in tensor categories

Let $\operatorname{Ind}(\mathcal{A})$ be the category of ind-objects of a tensor category $\mathcal{A}$ defined in section 2.5. The structure in this category suffices to define an internal commutative algebra theory.

Definition 5.33. An *algebra* in $\mathcal{A}$ is an object A of $\operatorname{Ind}(\mathcal{A})$ together with a multiplication $m : A \otimes A \to A$ and a unit $e : I \to A$ such that m is associative and commutative, and e is indeed a unit. A morphism of algebras in $\mathcal{A}$ is an arrow $f : A \to B$ in $\operatorname{Ind}(\mathcal{A})$ such that $f \circ m_A = m_B \circ (f \otimes f)$ and $f \circ e_A = e_B$. Denote the category of algebras in $\mathcal{A}$ by $\mathbf{Alg}_\mathcal{A}$.

Remark 5.34. An algebra in $\mathcal{A}$ is simply a commutative monoid object in the monoidal category $\operatorname{Ind}(\mathcal{A})$. The term "algebra" is chosen instead because this highlights the purpose of the construction. In particular, tensor categories closely resemble the category of vector spaces, and in this category we also call a commutative monoid "algebra".

Definition 5.35. If A is an algebra in $\mathcal{A}$, a *left A-module* is an object M in $\operatorname{Ind}(\mathcal{A})$ together with an action morphism $\rho : A \otimes M \to M$ such that $\rho \circ (e \otimes \mathrm{id}_M) = \mathrm{id}_M$ and $\rho \circ (m \otimes \mathrm{id}_M) = \rho \circ (\mathrm{id}_A \otimes \rho)$. A morphism of A-modules is an arrow $\varphi : M \to N$ such that $\varphi \circ \rho_M = \rho_N \circ (\mathrm{id}_A \otimes \varphi)$. The braiding in $\mathcal{A}$ can be used to view a left A-module as right A-module and vice versa, and I will always view A-modules as bimodules in this way.

Proposition 5.36. *For an algebra A in $\mathcal{A}$, the category $A\mathbf{Mod}$ of A-modules is abelian k-linear.*

Proof. The k-linearity follows from the fact that the hom-sets of $A\mathbf{Mod}$ are subsets of the hom-sets of $\operatorname{Ind}(\mathcal{A})$ which are closed under scalar multiplication and addition. Now note that the zero object of $\operatorname{Ind}(\mathcal{A})$ is an A-module with the trivial action, and thus also is a zero object in the category $A\mathbf{Mod}$. Further, if (M, ρ_M) and (N, ρ_N) are two A-modules, then the biproduct $M \oplus N$ of M and N in $\operatorname{Ind}(\mathcal{A})$ is again an A-module with action given by

$$A \otimes (M \oplus N) \simeq A \otimes M \oplus A \otimes N \xrightarrow{\rho_M \oplus \rho_N} M \oplus N,$$

and this is a biproduct of (M, ρ_M) and (N, ρ_N) in $A\mathbf{Mod}$. Now suppose $f : (M, \rho_M) \to (N, \rho_N)$ is a morphisms of A-modules. Then because $\operatorname{Ind}(\mathcal{A})$ is abelian, the underlying morphism $f : M \to N$ has a kernel $\ker(f)$ and cokernel $\operatorname{coker}(f)$, and the two following diagrams show how these have a canonical structure of A-modules.

$$\begin{array}{ccccc}
\ker(f) & \rightarrowtail & M & \xrightarrow{f} & N \\
\uparrow{\scriptstyle \rho_{\ker(f)}} & \nearrow{\scriptstyle \rho_M \circ a} & \uparrow{\scriptstyle \rho_M} & & \uparrow{\scriptstyle \rho_N} \\
A \otimes \ker(f) & \overset{}{\underset{a}{\rightarrowtail}} & A \otimes M & \xrightarrow[\mathrm{id}_A \otimes f]{} & A \otimes N
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes M & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes N & \twoheadrightarrow & A \otimes \operatorname{coker}(f) \\
\downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} & \searrow{\scriptstyle b \circ \rho_N} & \downarrow{\scriptstyle \rho_{\operatorname{coker}(f)}} \\
M & \xrightarrow[f]{} & N & \overset{}{\underset{b}{\twoheadrightarrow}} & \operatorname{coker}(f)
\end{array}$$

Note that I used the exactness of the tensor product to write $A \otimes \operatorname{coker}(f) \simeq \operatorname{coker}(\mathrm{id}_A \otimes f)$. These A-module structures make $\ker(f)$ and $\operatorname{coker}(f)$ the kernel and cokernel of f in $A\mathbf{Mod}$ respectively. It remains to show that every monic morphisms in $A\mathbf{Mod}$ is a kernel and every epimorphism a cokernel. To this end, note that a morphism in an additive category is monoic if and only if its kernel is zero, and epic if and only if its cokernel is zero. Now if we have a monomorphism f in $A\mathbf{Mod}$, then it must have a zero kernel, and since the underlying object of this kernel is the kernel in $\operatorname{Ind}(\mathcal{A})$ by construction, f must also have zero kernel there. Now $\operatorname{Ind}(\mathcal{A})$ is abelian, so f must be a kernel there. Then by the construction of kernels in $A\mathbf{Mod}$ it is clear that f must also be a kernel in $A\mathbf{Mod}$. One can make an analogous argumentation for epimorphisms. □

If A is an algebra in $\mathcal{A}$, we can define the notion of a tensor product of modules over A. If M and N are A-modules define the underlying object of $M \otimes_A N$ to be the coequalizer

$$M \otimes A \otimes N \rightrightarrows M \otimes N \twoheadrightarrow M \otimes_A N$$

where the two parallel arrows are given by the module action of A on M and N respectively. The morphism $A \otimes M \otimes A \otimes N \to M \otimes A \otimes N$ which is action of A on M descends to a morphism $A \otimes M \otimes_A \otimes N \to M \otimes_A N$, giving an A-module structure on $M \otimes_A N$.

Proposition 5.37. *This tensor product makes the category $A\mathbf{Mod}$ of A-modules into a symmetric monoidal category with unit given by $I_A = A$ as an A-module with action given by the multiplication.*

The proof of this statement is straight forward and can be found in [Mar09, prop. 1.2.15]. An A-module M is called *flat* if the functor $N \rightsquigarrow N \otimes_A M$ is exact. For any object X of $\operatorname{Ind}(\mathcal{A})$ we can define a free A-module $A \otimes X$ with action

$$A \otimes A \otimes X \xrightarrow{m \otimes \mathrm{id}_X} A \otimes X,$$

and free modules are always flat since $M \otimes_A (A \otimes X) \simeq A \otimes X$, and the tensor product in $\operatorname{Ind}(\mathcal{A})$ is exact.

Lemma 5.38. *Every A-module is a quotient of a flat A-module.*

Proof. Let M be an A-module. Then one can view $A \otimes M$ as a free A-module, and in this case $\rho : A \otimes M \to M$ is a morphism of A-modules. By the unit axiom, we have $\rho \circ (e \otimes \mathrm{id}_M) = \mathrm{id}_M$, so ρ is split epic and in particular, M is a quotient of $A \otimes M$. Since free modules are flat, this proves the claim. □

Another important class of flat modules are the ones that admit a dual.

Proposition 5.39. *In a symmetric monoidal category with dualizable object X, the functor $- \otimes X$ preserves all limits and colimits. In particular, it is exact and thus X is flat.*

Proof. In this case the functor $- \otimes X$ is both left- and right-adjoint to the functor $- \otimes X^\vee$. Since right-adjoint functors preserve limits and left-adjoint functors preserve colimits [ML78], this shows that $- \otimes X$ preserves all limits and colimits. □

Corollary 5.40. *Every dualizable modulue over some algebra A in $\mathcal{A}$ is flat.*

Definition 5.41. Let A be an algebra in $\mathcal{A}$. An A-algebra B is an algebra in $\mathcal{A}$ together with a morphism of algebras $f : A \to B$. Denote the category of A-algebras by $\mathbf{Alg}_A$.

We can define extension and restriction of scalars functors in this setting. If B is an A-algebra and M a B-module, the morphism

$$\rho_M \circ (f \otimes \mathrm{id}_M) : A \otimes M \to B \otimes M \to M$$

makes M into an A-module, which I will denote by $M|_A$ (*restriction of scalars*). In turn, we can send an A-module M to the free B-module $B \otimes_A M =: M_B$ (*extension of scalars*), and this operation is left adjoint to the restriction of scalars. In particular, for M an A-module and N a B-module, the adjunction isomorphism $\mathrm{Hom}_{B\mathbf{Mod}}(M_B, N) \xrightarrow{\simeq} \mathrm{Hom}_{A\mathbf{Mod}}(M, N|_A)$ and its inverse are given by

$$(u : M_B \to N) \mapsto (M \simeq A \otimes_A M \xrightarrow{f \otimes \mathrm{id}_M} B \otimes_A M \xrightarrow{u} N), \tag{6}$$

$$(v : M \to N) \mapsto (M_B = B \otimes_A M \xrightarrow{\mathrm{id}_B \otimes v} B \otimes_A N \simeq N). \tag{7}$$

Note that "extension of scalars" is a strong monoidal functor:

$$\begin{aligned}(M \otimes_A M')_B = B \otimes_A (M \otimes_A M') &\simeq (B \otimes_B B) \otimes_A (M \otimes_A M') \\ &\simeq (B \otimes_A M) \otimes_B (B \otimes_A M') = M_B \otimes_B M'_B\end{aligned}$$

and $A_B = B \otimes_A A \simeq B$.

This adjunction gives rise to an alternative way to view the category $\mathrm{Ind}(\mathcal{A})$. The isomorphism $I \otimes I \xrightarrow{\simeq} I$ makes I into an algebra with the identity as the unit, and every object X of $\mathrm{Ind}(\mathcal{A})$ admits a unique module structure over this algebra given by the isomorphism $I \otimes X \xrightarrow{\simeq} X$. Every morphism in $\mathrm{Ind}(\mathcal{A})$ is automatically a morphism of I-modules, and thus we have an equivalence

$$\mathrm{Ind}(\mathcal{A}) \simeq I\mathbf{Mod}.$$

If A is an algebra in $\mathcal{A}$, the unit diagram shows that $e : I \to A$ is indeed a morphism of algebras, i.e. the diagram

$$\begin{array}{ccc} I \otimes I & \xrightarrow{e \otimes e} & A \otimes A \\ \simeq \downarrow & & \downarrow m \\ I & \xrightarrow{e} & A \end{array}$$

commutes. Since I is a simple object of $\mathrm{Ind}(\mathcal{A})$ [EGNO15, Thm. 4.3.8], this morphism must have a trivial kernel or be the zero morphism, and in the later case we have $A = 0$. Using this, every algebra can be seen as an I-algebra, and the extension of scalars functor $I\mathbf{Mod} \to A\mathbf{Mod}$ is the exact functor $M \mapsto A \otimes M$.

The symmetric algebra. We can define the categorical analog to the symmetric algebra of a vector space. Let X be an object in $\mathrm{Ind}(\mathcal{A})$. In the last section we have seen how the symmetric group acts on the tensor powers of such an object.

Definition 5.42 (Symmetric power)**.** The *symmetric n-power* of X is the cokernel

$$X^{\otimes n} \xrightarrow{\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma} X^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n X.$$

If $f : X \to Y$ is a morphism in $\mathrm{Ind}(\mathcal{A})$, by the universal property of the cokernel we have an unique morphism $f^n : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ making the following diagram commute.

$$\begin{array}{ccccc}
X^{\otimes n} & \xrightarrow{\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma} & X^{\otimes n} & \twoheadrightarrow & \mathrm{Sym}^n X \\
\downarrow{\scriptstyle f^{\otimes n}} & & \downarrow{\scriptstyle f^{\otimes n}} \;\searrow{\scriptstyle a \circ f^{\otimes n}} & & \downarrow{\scriptstyle f^n} \\
Y^{\otimes n} & \xrightarrow[\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma]{} & Y^{\otimes n} & \xrightarrow{a} & \mathrm{Sym}^n Y
\end{array}$$

In this way, Sym^n becomes an endofunctor $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{A})$.

The interaction between different symmetric powers is very similar to the vector space case. In particular [Bra14, Prop. 4.4.4],

Proposition 5.43. *The tensor multiplication $X^{\otimes p} \otimes X^{\otimes q} \to X^{\otimes (p+q)}$ lifts to an epimorphism*

$$\mathrm{Sym}^p X \otimes \mathrm{Sym}^q X \to \mathrm{Sym}^{p+q} X.$$

The induced morphism

$$\bigoplus_{p+q=n} \mathrm{Sym}^p X \otimes \mathrm{Sym}^q Y \to \mathrm{Sym}^n(X \oplus Y)$$

is an isomorphism.

The symmetric powers assemble into an algebra in $\mathcal{A}$. Note that $\mathrm{Sym}^0 X = I$ for all objects X.

Definition 5.44 (Symmetric algebra)**.** For X an object in $\mathrm{Ind}(\mathcal{A})$, define the *symmetric algebra* over X as

$$\mathrm{Sym}(X) := \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \mathrm{Sym}^n X.$$

With unit $I = \mathrm{Sym}^0 X \hookrightarrow \mathrm{Sym} X$ and multiplication induced by the morphisms $\mathrm{Sym}^p X \otimes \mathrm{Sym}^q X \to \mathrm{Sym}^{(p+q)} X$ of proposition 5.43, this is indeed an algebra in $\mathcal{A}$, which is immediately clear since the multiplication morphisms inherit commutativity and associativity from the tensor multiplication. We can view Sym as a functor $\mathrm{Ind}(\mathcal{A}) \to \mathbf{Alg}_{\mathcal{A}}$.

The role of this functor is to provide a universal algebra over every object of $\mathrm{Ind}(\mathcal{A})$. To make this more precise,

Proposition 5.45. *The functor* $\mathrm{Sym} : \mathrm{Ind}(\mathcal{A}) \to \boldsymbol{Alg}_{\mathcal{A}}$ *is left-adjoint to the forgetful functor* $U : \boldsymbol{Alg}_{\mathcal{A}} \to \mathrm{Ind}(\mathcal{A})$.

Proof. Let A be some algebra in $\mathcal{A}$, and $f : X \to A$ a morphism in $\mathrm{Ind}(\mathcal{A})$. For any $n > 0$ the multiplication $m : A \otimes A \to A$ induces a morphism $A^{\otimes n} \to A$, and for $n = 0$ the unit is a morphism $I = A^{\otimes 0} \to A$. Commutativity of the multiplication in A ensures that these morphisms lift to morphisms $\mathrm{Sym}^n A \to A$, and then using f we get morphisms

$$\mathrm{Sym}^n X \xrightarrow{f^n} \mathrm{Sym}^n A \to A,$$

and hence a morphism $\tilde{f} : \mathrm{Sym} X \to A$. By construction, this is the unique morphism of algebras such that $(X = \mathrm{Sym}^1 X \hookrightarrow \mathrm{Sym} X \xrightarrow{\tilde{f}} A) = f$. □

We can repeat the whole construction for the category of modules over some algebra (A, e, m) in $\mathcal{A}$. For M an A-module, define the symmetric n-power over A as the cokernel

$$M^{\otimes_A n} \xrightarrow{\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma} M^{\otimes_A n} \twoheadrightarrow \mathrm{Sym}^n_A M.$$

This symmetric power over an algebra preserves free modules, i.e. $\mathrm{Sym}^n_A(A \otimes X) \simeq A \otimes \mathrm{Sym}^n(X)$. Therefore, the symmetric power of an A-module (M, ρ) is again an A-module with action ρ^n. Again, these symmetric powers assemble into an algebra, the symmetric A-algebra of M,

$$\mathrm{Sym}(M) := \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \mathrm{Sym}^n_A M$$

with unit $I \xrightarrow{e} A = \mathrm{Sym}^0_A M \hookrightarrow \mathrm{Sym}_A M$ and $A = \mathrm{Sym}^1_A \hookrightarrow \mathrm{Sym}_A M$. As before, the functor Sym_A is left-adjoint to the forgetful functor $U : \mathbf{Alg}_A \to \mathbf{Mod}_A$.

Proposition 5.46. *The symmetric powers preserve flatness, i.e. if M is a flat A-module, then $\mathrm{Sym}^n_A(M)$ also is a flat A-module for all n.*

Proof. As a left-adjoint, the symmetric algebra functor preserves colimits, and since it is the direct sum over the tensor powers, these must preserve colimits too. Further, we have already seen that the symmetric powers preserve freeness, and thus by Lazard's Theorem [Lur17, Thm. 7.2.2.15] they also preserve flatness. □

The exterior algebra. Analogously to the symmetric powers, we also have *exterior n-powers* $\Lambda^n X$ of an object X of $\mathrm{Ind}(\mathcal{A})$ given as the cokernel

$$X^{\otimes n} \xrightarrow{\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma} X^{\otimes n} \twoheadrightarrow \Lambda^n X$$

This can be made into an endofunctor of $\mathrm{Ind}(\mathcal{A})$ in the same way as the symmetric algebra, and proposition 5.43 holds when exchanging all symmetric powers by exterior powers [Bra14, Prop. 4.4.4]. These exterior powers can also be assembled into an algebra.

Definition 5.47. The exterior algebra of an object X in $\mathrm{Ind}(\mathcal{A})$ is given by

$$\Lambda(X) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \Lambda^n X.$$

Its unit is given by the fact that $\Lambda^0 X = I$, and its multiplication is induced by the tensor multiplication $X^p \otimes X^q \to X^{p+q}$. Note however that this "algebra" is non-commutative, and thus not formally an algebra by our definition.

We also have exterior powers and an exterior algebra in the category of modules over some algebra A in $\mathcal{A}$ just as in the symmetric case.

5.5 Proof of Deligne's theorem

To prove that every tensor category of subexponential growth admits a super fiber functor, we first need to translate the condition of subexponential growth into something more easy to handle. To this end, note

Proposition 5.48. *For an object X of a tensor category $\mathcal{A}$ the following are equivalent*
(i) There is a Schur functor annihilating X, i.e. there is a partition λ such that $S_\lambda(X) = 0$.
(ii) X is of subexponential growth (see definition 5.19).

It is easy to show that condition (ii) implies condition (i). Suppose that for an object X all Schur functors are non-zero, so in particular $\dim S_\lambda(X) > 0$ for all partitions λ. Then by $X^{\otimes n} \simeq \bigoplus_\lambda V_\lambda \otimes S_\lambda(X)$ (see theorem 5.26) and the properties of the dimension we have

$$\operatorname{length}(X^{\otimes n}) \geqslant \sum_{|\lambda|=n} \dim V_\lambda \geqslant \left(\sum_{|\lambda|=n} (\dim V_\lambda)^2 \right)^{\frac{1}{2}} = (n!)^{\frac{1}{2}},$$

since for a finite group G, every irreducible representation V appears in the regular representation $\dim V$ times [FH04]. But $(n!)^{1/2}$ grows faster than any power, and thus X cannot be of subexponential growth.

The other direction will follow from Deligne's theorem, and I will state the proof later in the section.

Proposition 5.49. *The collection of objects of a tensor category satisfying the conditions of proposition 5.48 is stable under direct sums, tensor products, duals, extensions*[10] *and subquotients.*

I will split the proof of proposition 5.49 into two lemmas. Let $\mathcal{A}$ be a k-linear abelian symmetric monoidal category with bilinear and exact tensor product.

Lemma 5.50. *The objects of $\mathcal{A}$ annihilated by at least one Schur functor is stable under direct sums and tensor products.*

Proof. Suppose $S_\mu(X) = S_\nu(Y) = 0$ and let $p, q, r, s \geqslant 0$ be integers such that $(p+1, q+1) \in [\mu']$ implies $[\mu] \subset [\mu']$ and $(r+1, s+1) \in [\nu']$ implies $[\nu] \subset [\nu']$. If λ is a partition satisfying $(p+r+1, q+s+1) \in [\lambda]$, we have by 5.27

$$S_\lambda(X \oplus Y) = \bigoplus_{|\mu'|+|\nu'|=|\lambda|} (S_{\mu'}(X) \otimes S_{\nu'}(Y))^{\oplus[\lambda:\mu',\nu']},$$

and if for some μ', ν' we have $[\lambda : \mu', \nu'] \neq 0$, corollary 5.31 implies that $(p+1, q+1) \in [\mu']$ or $(r+1, s+1) \in [\nu']$. By assumption, this yields $[\mu] \subset [\mu']$ or $[\nu] \subset [\nu']$, and by corollary 5.28 we get $S_\mu(X) = 0$ or $S_\nu(Y) = 0$. This shows stability under direct sums. For the tensor product, choose $(pr + qs + 1, ps + qr + 1) \in [\lambda]$ and proceed analogously. ☐

Lemma 5.51. *All objects of $\mathcal{A}$ annihilated by at least one Schur functor are of finite length, and the family of those objects is stable under subquotients and extensions.*

Proof. If Y is a subobject of X, then by exactness $Y^{\otimes n}$ is a subobject of $X^{\otimes n}$, and thus

$$\operatorname{Hom}(V_\lambda, Y^{\otimes n}) \hookrightarrow \operatorname{Hom}(V_\lambda, X^{\otimes n})$$

is an inclusion of a subobject. The functor "invariant under the action of $\mathfrak{S}_n$" is exact: If $k[\mathfrak{S}_n]$ again denotes the group algebra, this functor is the same as $\operatorname{Hom}_{k[\mathfrak{S}_n]\mathbf{Mod}}(I, -)$, where I is

[10] An extension of an object X by another object X' is a short exact sequence $0 \to X' \to X'' \to X \to 0$.

equipped with the trivial action. But this is clearly exact, since I is a direct factor of the free module $k[\mathfrak{S}_n]$ and thus projective. Therefore $S_\lambda(Y)$ is a subobject of $S_\lambda(X)$. Dualy the same argumentation shows that if Z is a quotient object of X, then $S_\lambda(Z)$ is a quotient of $S_\lambda(X)$, thus proving stability under subquotients.
Now let X be an extension of X' by X'', i.e. we have a short exact sequence

$$0 \to X'' \to X \to X' \to 0.$$

Then we obtain a filtration $X \supset X'' \supset 0$ of X with associated grading $X' \oplus X''$, and by the exactness of the tensor product we also get a filtration

$$X^{\otimes n} \supset X'' \otimes X^{\otimes(n-1)} \supset \cdots \supset (X'')^{\otimes n} \supset 0$$

of $X^{\otimes n}$ with associated grading $(X' \oplus X'')^{\otimes n}$ for all $n \in \mathbb{N}$. As this filtration is $\mathfrak{S}_n$-equivariant, this in turn yields a filtration of $S_\lambda(X)$ with associated grading $S_\lambda(X' \oplus X'')$ for every partition λ of n, showing that stability under extension follows directly from stability under direct sums. We can repeat this procedure with any finite filtration F or X, obtaining a filtration of $S_\lambda(X)$ with associated grading $S_\lambda(\mathrm{gr}_F(X))$ where $\mathrm{gr}_F(X)$ is the grading of F. Now by rigidity, the tensor product of two non-zero objects is non-zero, and therefore if $\mathrm{gr}^i_F(X) \neq 0$ for $i = 1, \ldots, n$, then $S_\lambda(\mathrm{gr}_F(X))$ contains their tensor product which is non-zero, and hence $S_\lambda(X) \neq 0$. This shows that $S_\langle(X) = 0$ must entail that $\mathrm{length}(X) < n$. □

Proof of proposition 5.49. The stability under taking duals follows directly from the fact that the Schur functor of a dual object is the dual object of the Schur functor of the original object, since this implies that $S_\lambda(X) = 0$ entails $S_\lambda(X^\vee) = 0$. The proposition now follows directly from lemma 5.50 and 5.51. □

If $\mathcal{A}$ is a tensor category with subexponential growth, in particular all its objects are of finite length, and we can use the following property simplifying the handling of the hom-spaces.

Proposition 5.52. *If all objects of a tensor category $\mathcal{A}$ are of finite length, then all its hom-spaces are finite dimensional.*

Proof. We have an internal hom given by $[X, Y] = Y \otimes X^\vee$, so in particular we have the tensor-hom adjunction

$$\mathrm{Hom}_{\mathcal{A}}(Z, [X, Y]) \simeq \mathrm{Hom}_{\mathcal{A}}(Z \otimes X, Y)$$

for some objects X, Y, Z of $\mathcal{A}$. In particular, choosing $Z = I$, we see that $\mathrm{Hom}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(I, [X, Y])$, and thus

$$\mathrm{Hom}_{\mathcal{A}}(I^{\oplus n}, [X, Y]) = \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathcal{A}}(I, [X, Y]) = \mathrm{Hom}_{\mathcal{A}}(X, Y)^{\oplus n}.$$

This shows that a morphism $f : I^{\oplus n} \to [X, Y]$ is given by n morphisms $f_i : X \to Y$, and since I is simple, the kernel of a morphism $I \to [X, Y]$ is either zero or I, so f is a monomorphism if and only if all the f_i are linearly independent. Now since $[X, Y]$ has finite length, it can only have finitely many factors of I, and so there is a maximal such n, showing that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional. □

We can now start the actual proof of Deligne's theorem 5.20. For the rest of this section, let $\mathcal{A}$ denote a tensor category of subexponential growth; so in particular all its objects are of finite length and for every object there is a Schur functor annihilating it. The first step is to find a more general fiber functor.

Definition 5.53. Let R be a (possibly infinite-dimensional) k-superalgebra. An R-fiber functor is a fiber functor from $\mathcal{A}$ into the category of R-modules.

The key result for the proof of theorem 5.20 is the following lemma.

Lemma 5.54. *Let M be a dualizable module over a non-zero algebra A in $\mathcal{A}$. There exists a non-zero A-algebra B such that the B-module I_B is a direct factor of M_B if and only if* $\mathrm{Sym}^n_A(M) \neq 0$ *for all n.*

Proof. For an A-algebra B, denote by $\mathrm{Fact}(B)$ the collection of pairs of morphisms of B-modules $(\alpha : I_B \to M_B, \beta : M_B \to I_B)$ such that $\beta \circ \alpha = \mathrm{id}_{I_B}$. Note that such α and β are exactly the data making I_B into a direct summand of M_B. We can make this into a functor from $\mathbf{Alg}_A$ to sets in the following way: Let B, C be A-algebras and $f : B \to C$ be a morphism making C a B-algebra. Now let $\alpha : I_B \to M_B$ and $\beta : M_B \to I_B$ be as above. Since $C \otimes_B B \simeq C$ and $C \otimes_B (C \otimes_A M) \simeq C \otimes_A M$, the extension of scalars functor from $\mathbf{Alg}_B$ to $\mathbf{Alg}_C$ sends these α and β to morphisms $\tilde{\alpha} : I_C \to M_C$ and $\tilde{\beta} : M_C \to I_C$, which by functorality also satisfy $\tilde{\beta} \circ \tilde{\alpha} = \mathrm{id}_{I_C}$.

Clearly, there exists an A-algebra B such that I_B is a direct factor of M_B if and only if this functor is non-trivial. The idea of the proof now is to compute a representing object (B_0, α_0, β_0) of this functor, and showing that it is non-zero if and only if all the symmetric A-powers of M are non-zero. More concretely, we want B_0 and $(\alpha_0, \beta_0) \in \mathrm{Fact}(B_0)$ such that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_A}(B_0, B) &\to \mathrm{Fact}(B) \\ f &\mapsto (\mathrm{ext}^B_{B_0}\alpha_0, \mathrm{ext}^B_{B_0}) \end{aligned}$$

is a natural isomorphism, where $\mathrm{ext}^B_{B_0}$ denotes the extension of scalars functor from B_0 to B which uses f to implement B as a B_0 algebra. To this end, let B be some A-algebra.

- A morphisms $\beta : M_B \to I_B$ of B-modules corresponds via the extension/restriction of scalars adjunction to a morphisms of A-modules $v : M \to I_B = B$. By proposition 5.45, this is in turn equivalent to a morphism of A-algebras

$$v_{\mathrm{alg}} : \mathrm{Sym}_A(M) \to B.$$

- A morphism $\alpha : I_B \to M_B$ of B-modules corresponds via the extension/restriction adjunction to a morphism of A-modules $u' : A \to B \otimes_A M$. We can make this into a morphism $u : M^\vee \to B$ by

$$M^\vee \simeq M^\vee \otimes_A A \xrightarrow{\mathrm{id}_{M^\vee} \otimes u'} M^\vee \otimes_A (B \otimes_A M) \simeq B \otimes_A M^\vee \otimes_A M \xrightarrow{\mathrm{ev}} B \otimes_A A \simeq B,$$

  and this indeed gives a one-to-one correspondence between morphisms $A \to B \otimes_A M$ and $M^\vee \to B$ with inverse sending $f : M^\vee \to B$ to

$$A \xrightarrow{\mathrm{coev}} M \otimes_A M^\vee \simeq M^\vee \otimes_A M \xrightarrow{f \otimes \mathrm{id}_M} B \otimes_A M.$$

  Again by proposition 5.45, $u : M^\vee \to B$ corresponds to a morphism of A-algebras

$$u_{\mathrm{alg}} : \mathrm{Sym}_A(M^\vee) \to B.$$

Let $\varphi : A \to B$ be the morphism making B into an A-algebra. By equation (6), the extension/restriction of scalars adjunction sends the identity on I_B to

$$(A \simeq A \otimes_A A \xrightarrow{\varphi \otimes \mathrm{id}_A} B \otimes_A A \xrightarrow{\mathrm{id}_{I_B}} B \otimes_A A \simeq B) = \varphi.$$

Thus, the condition that $\beta \circ \alpha = \mathrm{id}_{I_B}$ is equivalent to the restriction of $\beta \circ \alpha$ to A-algebras being the morphism φ making B an A-algebra. Expressing the restriction of $\beta \circ \alpha$ in terms of u and v using equations (6) and (7), we get

$$\begin{aligned}(I_A \xrightarrow{\mathrm{coev}} M \otimes_A M^\vee \simeq M^\vee \otimes_A M \xrightarrow{u \otimes \mathrm{id}_M} B \otimes_A M \xrightarrow{\mathrm{id}_B \otimes v} B \otimes_A B \xrightarrow{m_B} B) \\ = (I_A \xrightarrow{\mathrm{coev}} M \otimes_A M^\vee \simeq M^\vee \otimes M \xrightarrow{u \otimes v} B \otimes_A B \xrightarrow{m_B} B),\end{aligned}$$

which is just the product $u \cdot v = m_B \circ u \otimes v$ applied to the image of coev. This morphism $u \cdot v$ is again equivalently given by the A-algebra morphism

$$v_{\mathrm{alg}} \cdot u_{\mathrm{alg}} : \mathrm{Sym}_A(M) \otimes \mathrm{Sym}_A(M^\vee) \to B.$$

Note that since this is a morphism of A-algebras, it must be equal to φ in the zero component, i.e. $(A \simeq A \otimes_A A = \mathrm{Sym}^0_A(M) \otimes_A \mathrm{Sym}^0_A(M^\vee) \xrightarrow{v_{\mathrm{alg}} \cdot u_{\mathrm{alg}}} B) = \varphi$. As a result of this discussion, giving morphisms α and β making I_B a direct factor of M_B is equivalent to giving a morphism of A-algebras

$$\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) \to B$$

such that its composition with $\mathrm{coev} : I_A \to M \otimes_A M^\vee = \mathrm{Sym}^1_A(A) \otimes_A \mathrm{Sym}^1_A(M^\vee)$ and $\iota_A : I_A \xrightarrow{\simeq} A \otimes_A A = \mathrm{Sym}^0_A(M) \otimes_A \mathrm{Sym}^0_A(M^\vee)$ yields the same morphism. Therefore, the universal A-algebra B_0 must be given as the coequalizer

$$I_A \underset{\mathrm{coev}}{\overset{\iota_A}{\rightrightarrows}} \mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) \xrightarrow{b_0} B_0.$$

Since $\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) = \bigoplus_{p,q \in \mathbb{Z}_{\geqslant 0}} \mathrm{Sym}^p_A(M) \otimes_A \mathrm{Sym}^q_A(M^\vee)$ is a direct sum, the morphism b_0 already is determined by its action on all $\mathrm{Sym}^p_A(M) \otimes_A \mathrm{Sym}^q_A(M^\vee)$. We can view the coevaluation as morphism

$$\mathrm{Sym}^p_A(M) \otimes_A \mathrm{Sym}^q_A(M^\vee) \to \mathrm{Sym}^{p+1}_A(M) \otimes_A \mathrm{Sym}^{q+1}_A(M^\vee)$$

by composing $\mathrm{Sym}^p_A(M) \otimes_A \mathrm{Sym}^q_A(M^\vee) \simeq A \otimes_A \mathrm{Sym}^p_A(M) \otimes_A \mathrm{Sym}^q_A(M^\vee)$ with coev and the multiplication map. Call this morphism $\delta_{p,q}$. Then the restrictions of b_0 to the direct summands have to make the following diagrams commute

$$\begin{array}{ccc} \mathrm{Sym}^n_A(M) \otimes_A \mathrm{Sym}^{n+a}_A(M^\vee) & \xrightarrow{\delta_{p,q}} & \mathrm{Sym}^{n+1}_A(M) \otimes_A \mathrm{Sym}^{n+a+1}_A(M^\vee) \\ & {\scriptstyle b_0} \searrow \quad \swarrow {\scriptstyle b_0} & \\ & B_0 & \end{array}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathbb{Z}$, and since $\mathrm{Sym}_A M$ and $\mathrm{Sym}_A M^\vee$ are flat, B_0 is the direct limit

$$B_0 = \bigoplus_{a \in \mathbb{Z}} \varinjlim_{n \in \mathbb{Z}_{\geqslant 0}} \mathrm{Sym}^n_A(M) \otimes_A \mathrm{Sym}^{n+a}_A(M^\vee) \tag{8}$$

with the coevaluation as transition morphisms. Since $b_0 \circ \iota_A = b_0 \circ \mathrm{coev}$ and b_0 must send the unit of $\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee)$ to the unit of B_0, we see that this unit of B_0 must be the direct limit of

$$(\mathrm{coev}^n : I \to \mathrm{Sym}^n_A M \otimes_A \mathrm{Sym}^n_A M^\vee) = \delta_{n-1,n-1} \circ \cdots \circ \delta_{0,0} \circ e,$$

where $e : I \to A$ is the unit of A. It therefore suffices to show that $\mathrm{coev}^n \neq 0$ for all n if and only if $\mathrm{Sym}^n_A M \neq 0$ for all n. But this is trivial, since coev^n is just the coevaluation map of the duality

$$\mathrm{Sym}^n_A(M^\vee) \simeq (\mathrm{Sym}^n_A(M))^\vee.$$

□

The reason why more tensor categories are categories of super representations than categories of ordinary representations lies in the fact that it is much easier to have a well defined "superdimension" $p|q$ than an ordinary dimension n. To make this more precise,

Definition 5.55. Let $\bar{I}$ be an object of $\mathcal{A}$ such that $\bar{I} \otimes \bar{I} \simeq I$ and the braiding $\bar{I} \otimes \bar{I} \to \bar{I} \otimes \bar{I}$ is the multiplication by (-1). This object is simple since I is simple. Define a functor $F : \mathbf{sVect}_k \to \mathcal{A}$ sending a super vector space $V = V_0 \oplus V_1$ to

$$F(V) = (V_0 \otimes I) \oplus (V_1 \otimes \bar{I})$$

and sending a morphism $f : V \to W$ to $f_0 \otimes \mathrm{id}_I \oplus f_1 \otimes \mathrm{id}_{\bar{I}}$.

The conditions on $\bar{I}$ ensure that it mimics the behavior of the super vector space $V = 0 \oplus k$.

Proposition 5.56. *This functor F is strong monoidal, and if $\langle I, \bar{I} \rangle$ denotes the full subcategory of direct sums of copies of I and $\bar{I}$, $F : \boldsymbol{sVect}_k \to \mathcal{A}$ is an equivalence of categories.*

Proof. The isomorphism $\bar{I} \otimes \bar{I} \simeq I$ induces a natural isomorphism

$$\begin{aligned} F(V) \otimes F(W) &= (V_0 \otimes I \oplus V_1 \otimes \bar{I}) \otimes (W_0 \otimes I \oplus W_1 \otimes \bar{I}) \\ &\simeq (V_0 \otimes W_0 \otimes I \otimes I) \oplus (V_1 \otimes W_1 \otimes \bar{I} \otimes \bar{I}) \\ &\quad \oplus (V_0 \otimes W_1 \otimes I \otimes \bar{I}) \oplus (V_1 \otimes W_0 \otimes I \otimes \bar{I}) \\ &\simeq (V_0 \otimes W_0 \oplus V_1 \otimes W_1) \otimes I \oplus (V_0 \otimes W_1 \oplus V_1 \otimes W_0) \otimes I \otimes \bar{I} \\ &= F(V \otimes W), \end{aligned}$$

and we clearly have $F(k) = k \otimes I \simeq I$. This proves the first part. For the second part, note that if $\{e_i\}_{i \in I}$ is a basis of a vector space V, then $V \otimes X \simeq \bigoplus_I X$, so $F(V)$ indeed lies in $\langle I, \bar{I} \rangle$ for each super vector space V. F is faithful since $f \neq g : V \to W$ means that $f_0 \neq g_0$ or $f_1 \neq g_1$. It is also full, since the simplicity of I and $\bar{I}$ ensures that every morphism $f : I^{\oplus p} \oplus \bar{I}^{\oplus q} \to I^{\oplus r} \oplus \bar{I}^{\oplus s}$ must have components f_i^j which are either zero or the identity. Therefore, such a morphism can only commute factors of I and $\bar{I}$, or annihilate them, and all these operators are possible with morphisms of the form $g \otimes \mathrm{id}_I \oplus h \otimes \mathrm{id}_{\bar{I}}$ for linear g and h. That F is essentially surjective is clear by the definition of $\langle I, \bar{I} \rangle$. □

We can WOLOG assume that the category $\mathcal{A}$ admits such an object $\bar{I}$: If there is no such object, replace $\mathcal{A}$ by the category of $\mathbb{Z}_2$-graded objects of $\mathcal{A}$. In section 5.1 we have seen that this category also is a tensor category, and it clearly inherits the condition of subexponential growth. Then the unit I seen as odd element is such a $\bar{I}$, and since $\mathcal{A}$ can be viewed as the subcategory of even objects of $\mathcal{A}^{\mathbb{Z}_2}$, an R-fiber functor for $\mathcal{A}^{\mathbb{Z}_2}$ also yields one for $\mathcal{A}$. The next proposition shows that we can expect every object of $\mathcal{A}$ to have a well defined "super-dimension" after some extension of scalars.

Proposition 5.57. *For an object X of $\mathcal{A}$ the following are equivalent.*

(i) There exist non-negative integers p and q such that $X_A \simeq (I^{\oplus p} \oplus \bar{I}^{\oplus q})_A$ for some non-zero algebra A.

(ii) There is a Schur functor annihilating X.

Proof. "$(i) \Rightarrow (ii)$" For every partition λ, $S_\lambda(X)_A$ is isomorphic to $S_\lambda(I^{\oplus p} \oplus \bar{I}^{\oplus q})_A$, and since $\langle I, \bar{I} \rangle \sim \mathbf{sVect}_k$, we can use proposition 5.30 to deduce that there is some λ such that $S_\lambda(I^{\oplus p} \oplus \bar{I}^{\oplus q})_A = 0$. Since A is non-zero, this also entails $S_\lambda(X) = 0$.
"$(ii) \Rightarrow (i)$" Suppose that for some non-zero algebra A we get

$$X_A \simeq I_A^{\oplus r} \oplus \bar{I}_A^{\oplus s} \oplus R$$

with R some A-module. Then R is dualizable as direct factor of the dualizable A-module X_A. We consider three different cases:

(a) All the $\mathrm{Sym}^n_A(R)$ are non-zero. By lemma 5.54 there is a non-zero A-algebra B such that I_B is a direct factor of R_B. Then we get a new decomposition

$$X_B \simeq I_B^{\oplus r+1} \oplus \bar{I}_B^{\oplus s} \oplus R'$$

for some B-module R'.

(b) All the $\mathrm{Sym}^n_A(\bar{I} \otimes R)$ are non-zero. Again by lemma 5.54 we find an A-algebra B such that I_B is a direct factor of $(\bar{I} \otimes R)_B$, and a direct factor I_B of this object is equivalent to a direct factor $\bar{I}_B$ of R_B, so we get a new decomposition

$$X_B \simeq I_B^{\oplus r} \oplus \bar{I}_B^{\oplus s+1} \oplus R'$$

for some B-module R'.

(c) Non of the above. Since $\mathrm{Sym}^n_A(\bar{I} \otimes R) \simeq \bar{I}^{\otimes n} \otimes \Lambda^n_A R$, we then have n and m such that $\mathrm{Sym}^{n+1}_A R = \Lambda^{n+1}_A R = 0$. Let $k > mn$ and λ a partition of k. Since $[\lambda]$ contains a row of length greater than n or a column of length greater than m, and by corollary 5.28 we get $S_\lambda(R) = 0$. Since further by theorem 5.26

$$R^{\otimes k} \simeq \bigotimes_{|\lambda|=k} V_\lambda \otimes S_\lambda(R) = 0,$$

lemma 2.12 implies that $R = 0$, because R is dualizable as direct factor of a dualizable module. In this case we have a situation as in (i).

Starting with $A = I$, $r = s = 0$ and $R = X$ we can iteratively apply cases (a) and (b), and either reach case (c) at one point, or continue indefinitely. In the later case, X admits a direct factor $I^{\oplus p} \oplus \bar{I}^{\oplus q}$ after extension of scalars for arbitrarily large $p + q$. If this happens, for each n one can choose a partition λ of n and p, q such that $n < (p + 1)(q + 1)$, and by proposition 5.30, $S_\lambda(I^{\oplus p} \oplus \bar{I}^{\oplus q})$ is non-zero. But since $I^{\oplus p} \oplus \bar{I}^{\oplus q}$ is a direct factor of X after extension of scalars, $S_\lambda(I^{\oplus p} \oplus \bar{I}^{\oplus q})$ is a direct factor of $S_\lambda(X)$ after extension of scalars, which is in contradiction to $S_\lambda(X)$ being zero. This shows that we always reach case (c) and thus proves the claim. □

Recall the notion of an exact sequence from section 2.2. We need one more technical result for the proof of the existence of some R-fiber functor for $\mathcal{A}$. It can be found in [Del90, 7.14].

Lemma 5.58. *Every short exact sequence in $\mathcal{A}$ splits after some (non-trivial) extension of scalars.*

We are now finally in a position to show the existence of a super algebra R such that we find an R-fiber functor $\omega : \mathcal{A} \to R\mathbf{Mod}$.

Proposition 5.59 (Existence of R-fiber functor)**.** *If every object of a tensor category $\mathcal{A}$ is annihilated by some Schur functor, then there exists a non-trivial fiber functor from $\mathcal{A}$ to some possibly infinite dimensional super k-algebra.*

Proof. By proposition 5.57, for each isomorphism class $[X]$ in $\mathcal{A}$ there exists some non-zero algebra B such that $X_B \simeq (I^{\oplus p} \oplus \bar{I}^{\oplus q})_B$ for some p and q, and by lemma 5.58, for each isomorphism class $[\Sigma]$ of short exact sequences in $\mathcal{A}$ there exists some non-zero algebra C such that Σ_C is split. Let A be the filtered colimit of the tensor products of finitely many of those algebras B and C. This colimit exists by construction of the ind-category. The algebra A is non-zero, and after extension of scalars to A every object of $\mathcal{A}$ is a sum of copies of I and $\bar{I}$, and every short exact sequence splits.

We can identify the category of super vector spaces with the full subcategory $\langle I, \bar{I}\rangle$ of $\mathcal{A}$, and we can further identify $\mathrm{Ind}\langle I, \bar{I}\rangle$ with the category of possibly infinite dimensional super vector spaces. For X an object of $\mathrm{Ind}(\mathcal{A})$, denote by $\rho(X)$ the largest subobject of X which is an object of

$\mathrm{Ind}\langle I, \bar{I}\rangle$. This object $\rho(X)$ can be identified with the (possibly infinite dimensional) super vector space

$$\rho(X) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(I, X) \oplus \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(\bar{I}, X).$$

We can make this into a functor $\rho : \mathrm{Ind}\mathcal{A} \to \mathrm{Ind}\langle I, \bar{I}\rangle$ by sending a morphism $f : X \to Y$ to $f_\star \oplus f_\star : \rho(X) \to \rho(Y)$, where $f_\star(g) = f \circ g$. The multiplication $m : A \otimes A \to A$ induces a multiplication on $\rho(A)$ by

$$\rho(A) \otimes \rho(A) \hookrightarrow \rho(A \otimes A) \xrightarrow{\rho(m)} \rho(A),$$

and the unit $e : I \to A$ induces a unit $\rho(I) = \mathrm{Hom}_{\mathrm{Ind}\mathcal{A}}(I, I) \simeq k \to \rho(A)$ making $\rho(A)$ into a (possibly infinite dimensional) superalgebra. If M is an A-module, the action $A \otimes M \to M$ induces a morphism $\rho(A) \otimes \rho(M) \hookrightarrow \rho(A \otimes M) \to \rho(M)$ making $\rho(M)$ a $\rho(A)$-module. If M and N are two A-modules, the universal property of the coequalizer gives a unique morphism $\rho(M) \otimes_{\rho(A)} \rho(N) \to M \otimes_A N$ making the following diagram commute

$$\begin{array}{ccccc} \rho(M)\otimes\rho(A)\otimes\rho(N) & \longrightarrow & \rho(M)\otimes\rho(N) & \twoheadrightarrow & \rho(M)\otimes_{\rho(A)}\rho(N) \\ \downarrow & & \downarrow a & \overset{b\circ a}{\searrow} & \downarrow \\ M\otimes A\otimes N & \longrightarrow & M\otimes N & \overset{b}{\twoheadrightarrow} & M\otimes_A N \end{array}$$

where the vertical arrows are inclusion as subobject. Since $\rho(\rho(X)) = \rho(X)$ for all objects X, this morphism induces a morphism

$$\rho(M) \otimes_{\rho(A)} \rho(N) \to \rho(M \otimes_A N)$$

Now suppose that $M = A \otimes M_0$ with M_0 an object of $\langle I, \bar{I}\rangle$. Then $M_0 = I^{\oplus p} \oplus \bar{I}^{\oplus q}$, and $A \otimes M_0 = A \otimes I^{\oplus p} \oplus A \otimes \bar{I}^{\oplus q} \simeq X^{\oplus p} \oplus (X \otimes \bar{I})^{\oplus q}$, and similarly for $\rho(A) \otimes M_0$, so we get

$$\begin{aligned} \rho(M) = \rho(A \otimes M_0) &\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(I, A^{\oplus p} \oplus (A \otimes \bar{I})^{\oplus q}) \oplus \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(\bar{I}, A^{\oplus p} \oplus (A \otimes \bar{I})^{\oplus q}) \\ &= \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(I, A)^{\oplus p} \oplus \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(I, A \otimes \bar{I})^{\oplus q} \\ &\quad \oplus \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(\bar{I}, A)^{\oplus p} \oplus \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(\bar{I}, A \otimes \bar{I})^{\oplus q} \\ &= \rho(A)^{\oplus p} \oplus \rho(A \otimes \bar{I})^{\oplus q} = \rho(A) \otimes M_0. \end{aligned}$$

If in addition $N = A \otimes N_0$ with N_0 an object of $\langle I, \bar{I}\rangle$, then we have $M \otimes_A N \simeq A \otimes M_0 \otimes N_0$ and the following commutative diagram shows that the morphism $\rho(M) \otimes_{\rho(A)} \rho(N) \to \rho(M \otimes_A N)$ is indeed an isomorphism in this case.

$$\begin{array}{ccc} (\rho(A)\otimes M_0)\otimes_{\rho(A)}(\rho(A)\otimes N_0) & \xrightarrow{\simeq} & \rho(A)\otimes M_0\otimes N_0 \\ \simeq\downarrow & & \downarrow\simeq \\ \rho(M)\otimes_{\rho(A)}\rho(N) & \longrightarrow & \rho(M\otimes_A N) \end{array}$$

Now define a super k-algebra by $R := \rho(A)$, and let

$$\omega : \mathcal{A} \to R\mathbf{Mod} \tag{9}$$

be the functor sending an object X of $\mathcal{A}$ to the R-module $\rho(X_A)$. By construction, X_A must be of the form $A \otimes X_0$ with $X_0 \in \langle I, \bar{I}\rangle$, and thus we have an isomorphism $\omega(X) \otimes_R \omega(Y) \to \omega(X \otimes Y)$. Since also $\omega(I) = \rho(A) = R$, this shows that ω is a strong monoidal functor. By construction, ω is k-linear, and it is faithful because the Yoneda embedding and therefore ρ is. If Σ is an exact sequence in $\mathcal{A}$, then Σ_A is split, and since additive functors preserve split exactness, $\omega(\Sigma) = \rho(\Sigma_A)$ is also split exact, proving the exactness of ω. This finishes the proof. □

Existence of a super fiber functor over k. Deligne now uses some techniques from algebraic geometry to reduce the R-fiber functor to a k-fiber functor [Del02, Section 3 and 4]. I will state the result without proof.

Proposition 5.60 ([Del02, prop. 4.5])**.** *If $\mathcal{A}$ is a tensor category which admits a fiber functor over some non-zero super k-algebra, then $\mathcal{A}$ also admits a super fiber functor over k.*

We can now finish the proof of the main theorem, and also of proposition 5.48.

Proof of Deligne's theorem on tensor categories 5.20. If $\mathcal{A}$ satisfies condition (ii) of proposition 5.48, i.e. is of subexponential growth, it also satisfies condition (i) (see discussion below 5.48) and therefore we can use proposition 5.57 to find a fiber functor of $\mathcal{A}$ over some non-zero superalgebra R. Then by proposition 5.60, it also admits a super fiber functor over k. We can now use the recognition theorem for affine supergroups 5.17 to conclude that $\mathcal{A}$ is of the form $\mathbf{Rep}(G, p)$ for some affine supergroup G and $p \in G(k^{1|0})$. □

Proof of proposition 5.48. "$(i) \Rightarrow (ii)$" If every object of $\mathcal{A}$ is annihilated by some Schur functor, then by proposition 5.59 and 5.60 we have a super fiber functor $\omega : \mathcal{A} \to \mathbf{sVect}_k$ and if X is an object in $\mathcal{A}$ with $\omega(X)$ of dimension $p|q$, then because ω is faithful and exact, $X^{\otimes n}$ is of length at most $\text{length}(\omega(X)^{\otimes n}) = (p+q)^n$. □

6 Applications in quantum field theory

In modern physics and in particular in quantum field theory, representation theory plays an important role, since most physical symmetries are implemented in form of a representation of an algebraic object such as a group or a Lie algebra. In this section I will explore the use of Tannaka duality in the context of physical symmetries.

6.1 Symmetry groups

To begin with, we need to define what is meant by "physical symmetry". Informally speaking, a symmetry of a physical system is a transformation of the corresponding mathematical model that does not change the physics predicted by that model. The precise notion of a symmetry therefore depends on the mathematical model in use.

For example, in classical mechanics a symmetry is given by a change of the Lagrangian which leaves the stationary points of the action functional invariant. A good example are spacial symmetries like rotations that exploit geometrical properties of the physical system, like rotation invariance. In quantum mechanics a symmetry is an endomorphism of the projective space over the Hilbert space of states which leaves all the transition probabilities invariant. I will be mostly be concerned with symmetries in quantum field theory. Here, similar to classical mechanics, a symmetry is a transformation of the fields and the spacetime which maps solutions to the equation of motion to other solutions.

One distinguishes between two different kinds of symmetries. *Geometric symmetries* are linear endomorphisms of the underlying spacetime, and in the standard model of particle physics these endomorphism are given by the *Poincaré group*, or to be more precise, the universal cover of the identity component of the Poincaré group. The other kind of symmetry is called *internal symmetry*, and they are characterized by leaving the spacetime points invariant. These internal symmetries usually form a compact Lie group; in case of the standard model this is $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ [BH10].

There is a theorem due to S. Coleman and J. Mandula stating that under some reasonable physical assumptions, a quantum field theory whose geometric symmetry group contains the Poincaré group cannot have mixed internal and spacial symmetries, i.e. the full symmetry group is always given by a direct product of the geometric and internal symmetry group [CM67]. One way to circumvent this theorem is allowing the symmetries to form more general structure than a group, for example a supergroup. The corresponding physical theory exploring the consequences of this relaxation is called *supersymmetry*. The necessity of such a theory lies in the known phenomena that stay unexplained by the standard model, as well as deviations of the standard model from measurement in some points [Ell02]. Up until now, no experimental evidence for a supersymmetry has been found, but there are strong theoretical implications of their existence. I will elaborate on one of these in section 6.4.

6.2 Tannaka-Krein duality

To see the relevance of Tannaka duality for quantum field theory, we first need to adapt to the typical situation in physics. Here, one usually considers continuous complex representations of topological groups on Hilbert spaces, and often one in particular considers compact or locally compact topological groups. As explained in the introduction, a corresponding reconstruction theorem was given in the 1940's by T. Tannaka and M.G. Krein, which is also the reason for the naming "Tannaka duality". In particular, Tannaka proved that a compact topological group G is isomorphic to the group of self-conjugate natural endomorphisms of the forgetful functor $U : \mathbf{Rep}_{\mathbb{C}}(G) \to \mathbf{Vect}_{\mathbb{C}}$, where the conjugate of a natural transformation $\alpha : U \to U$ is defined as

$$\bar{\alpha}_V(x) = \overline{\alpha_{\bar{V}}(x)}$$

with $\bar{V}$ the conjugate representation to V. This statement along with a proof can be found in [JS91, section 1]. In view of the application of this theorem to algebraic quantum field theory, I will present a more modern formulation of Tannaka's classical result.

$\star$-Categories. The additional structure in the category of unitary representations on Hilbert spaces in comparison to linear representations over an unspecified field is the existence of a hermitian adjoint. To capture this structure categorically, we define

Definition 6.1. A $\star$-operation on an abelian $\mathbb{C}$-linear category is an assignment of a morphism $s^\star : Y \to X$ to every morphism $s : X \to Y$ such that this map is antilinear, $(s^\star)^\star = s$ and $s^\star \circ s = 0$ implies $s = 0$. A *$\star$-category* is a $\mathbb{C}$-linear abelian category equipped with a $\star$-operation. A *tensor $\star$-category* is a tensor category that is also a $\star$-category and satisfies $(s \otimes t)^\star = s^\star \otimes t^\star$. A functor F between $\star$-categories is $\star$-preserving if $F(s^\star) = F(s)^\star$ for all morphisms s.

In a Hilbert space, we call an operator A unitary if $A^\dagger A = AA^\dagger = \mathbb{1}$. Analogously, a morphism $s : X \to Y$ in a $\star$-category is called *unitary* if $s^\star \circ s = \mathrm{id}_X$ and $s \circ s^\star = \mathrm{id}_Y$. An endomorphism p of an object X is called projection of $p = p \circ p = p^\star$. We say that a $\star$-category *has subobjects* if for every object X and every projection $p : X \to X$ there is an $s : Y \to X$ with $s^\star \circ s = \mathrm{id}_Y$ and $s \circ s^\star = p$.

A braiding of a tensor $\star$-category is required to be unitary in the sense that its components are all unitary, and the isomorphisms associated to a strong monoidal functor of tensor $\star$-categories are also required to be unitary. The notion of duals is replaced by *conjugate objects* that model the complex conjugate of a representation.

Definition 6.2. Let $\mathbf{C}$ be a tensor $\star$-category, and X an object in $\mathbf{C}$. An object $\bar{X}$ is called *conjugate* of X if there are morphisms $r : I \to \bar{X} \otimes X$ and $\bar{r} : I \to X \otimes \bar{X}$ such that the morphisms

$$X \simeq I \otimes X \xrightarrow{\bar{r} \otimes \mathrm{id}_X} X \otimes \bar{X} \otimes X \xrightarrow{\mathrm{id}_X \otimes r^\star} X \otimes I \simeq X \quad \text{and}$$
$$\bar{X} \simeq I \otimes \bar{X} \xrightarrow{r \otimes \mathrm{id}_{\bar{X}}} \bar{X} \otimes X \otimes \bar{X} \xrightarrow{\mathrm{id}_{\bar{X}} \otimes \bar{r}^\star} \bar{X} \otimes I \simeq \bar{X}$$

are both the identity.

We can now make precise what we expect from a category that is similar to the category of finite dimensional Hilbert spaces $\mathcal{H}$**ilb**.

Definition 6.3. A $\mathbf{TC}^\star$ is a tensor $\star$-category with finite dimensional hom-sets, conjugates and subobjects. A $\mathbf{BTC}^\star$ is a $\mathbf{TC}^\star$ with unitary braiding, and a $\mathbf{STC}^\star$ is a $\mathbf{BTC}^\star$ whose braiding is in fact symmetric.

As expected, the category of finite dimensional Hilbert spaces is a $\mathbf{STC}^\star$ with the usual tensor category structure and conjugates given by the dual Hilbert space. Similarly, for a compact topological group, the category of finite dimensional unitary representations is an $\mathbf{STC}^\star$.

Definition 6.4. Let $\mathbf{C}$ be a tensor $\star$-category and X an object of $\mathbf{C}$. A conjugate $(\bar{X}, r, \bar{r})$ is called *standard* if for all $s \in \mathrm{End}_{\mathbf{C}}(X)$ we have

$$r^\star \circ \mathrm{id}_{\bar{X}} \otimes s \circ r = \bar{r}^\star \circ s \otimes \mathrm{id}_{\bar{X}} \circ \bar{r}.$$

In a $\mathbf{TC}^\star$ every object admits a standard conjugate [HM06, lem. A.37].

Just as dual objects in a rigid tensor category can be used to define a trace and dimension (see definition 2.18), conjugate objects can too. For $\mathbf{C}$ a $\mathbf{TC}^\star$, X an object of $\mathbf{C}$ and $(\bar{X}, r, \bar{r})$ a standard conjugate of X, define the trace of an endomorphism $s : X \to X$ as

$$\mathrm{tr}(s) = r^\star \circ \mathrm{id}_{\bar{X}} \otimes s \circ r \in \mathrm{End}_{\mathbf{C}}(I) \simeq \mathbb{C}.$$

The dimension of X is then defined as $\dim(X) = \mathrm{tr}(\mathrm{id}_X) = r^\star \circ r$.

Reconstruction theorem for compact groups. We can now formulate the reconstruction theorem of Tannaka and Krein in the language of $\star$-categories.

Definition 6.5. A $\star$-preserving fiber functor of a $\mathbf{STC}^\star$ $\mathbf{C}$ is a faithful functor of tensor $\star$-categories $\mathbf{C} \to \mathcal{H}$**ilb**.

Now assume that for some $\mathbf{STC}^\star$ $\mathbf{C}$ we are given a $\star$-preserving fiber functor E. Let $G_E \subset \mathrm{End}^{\otimes}(E)$ be the subset of unitary monoidal natural endomorphism of E. This clearly is a group under composition with neutral element the identity transformation, and if $g \in G_E$, every component g_X is a unitary operator in the Hilbert space $E(X)$, so we can identify G_E with a subset of $\prod_{X \in \mathrm{ob}(\mathbf{C})} \mathrm{U}(E(X))$, where $U(H)$ denotes the group of unitary operators of H. Since all the $E(X)$ are assumed to be of finite dimension, the unitary groups $U(E(X))$ are compact and by Tychonoff's theorem, the product over the objects of $\mathbf{C}$ is compact too. Since G_E is a closed subset, it is therefore a compact topological group.

We have a canonical action of G_E on all the Hilbert spaces $E(X)$ in form of unitary representations π_X, namely

$$\pi_X(g) = g_X$$

for $g \in G_E$.

Proposition 6.6. *Let $\boldsymbol{C}$ be a $\boldsymbol{STC^\star}$ with symmetric $\star$-preserving fiber functor E. There is a faithful symmetric strong monoidal $\star$-preserving functor $F : \boldsymbol{C} \to \boldsymbol{Rep}_{\mathcal{H}}(G_E)$*[11] *such that $K \circ F = E$ where $K : \boldsymbol{Rep}_{\mathcal{H}}(G_E) \to \mathcal{H}\boldsymbol{ilb}$ is the forgetful functor $(H, \pi) \rightsquigarrow H$.*

(Sketch). One defines $F(X) = (E(X), \pi_X)$ for X an object of $\mathbf{C}$, and $F(s) = E(s)$ for morphisms s. All the conditions that a symmetric strong monoidal $\star$-preserving functor has to satisfy are easily checked, and faithfulness follows directly from the faithfulness of E. □

A categorical proof of the main theorem can be found in [HM06, thm. B.6].

Theorem 6.7 (Tannaka-Krein duality)**.** *Let $\boldsymbol{C}$ be a $\boldsymbol{STC^\star}$ with symmetric $\star$-preserving fiber functor E. Then the functor F as in proposition 6.6 is an equivalence of symmetric tensor $\star$-categories.*

In other words, a compact group G can be recovered as group of unitary natural endomorphisms of the fiber functor of its category of unitary representation.

Recognition theorem for compact groups. The Tannaka reconstruction theorem 6.7 shows how to recover a compact group when its category of finite dimensional unitary representations is known. Krein additionally provided a criterion for a category to be of the form $\mathbf{Rep}_{\mathcal{H}}(G)$ for some compact group G. This too can be reformulated in the language of $\star$-categories.

Definition 6.8. Let $\mathbf{C}$ be a $\mathbf{BTC}^\star$, X an object of $\mathbf{C}$ and $(\bar{X}, r, \bar{r})$ a standard conjugate of X. The *twist* $\Theta(X)$ of X is defined by

$$\Theta(X) = r^\star \otimes \mathrm{id}_X \circ \mathrm{id}_{\bar{X}} \otimes \tau_{X,X} \circ r \otimes \mathrm{id}_X .$$

Lemma 6.9 ([HM06, lem. A.44])**.** *The twist of a $\boldsymbol{BTC^\star}$ $\boldsymbol{C}$ satisfies the following properties*

- *For a morphism $s : X \to Y$, one has $\Theta(Y) \circ s = s \circ \Theta(X)$, i.e. Θ is a natural transformation of the identity functor.*
- *$\Theta(X)$ is unitary for all objects X.*
- *If $\boldsymbol{C}$ is a $\boldsymbol{STC^\star}$, then $\Theta(X \otimes Y) = \Theta(X) \otimes \Theta(Y)$, i.e. Θ is a strong monoidal functor.*

In a $\mathbf{STC}^\star$ we have $\tau^\star_{X,X} = \tau^{-1}_{X,X} = \tau_{X,X}$, and therefore $\Theta(X^\star) = \Theta(X)$. Since by the above lemma $\Theta(X)$ is unitary, this yields $\Theta(X)^2 = \mathrm{id}_X$, and therefore in the case $\mathrm{End}_{\mathbf{C}}(X) = \mathbb{C}$ we get $\Theta(X) = \pm 1$.

Remark 6.10. This is the $\mathbf{TC}^\star$ variant of a ribbon category. The interpretation of such a twist is best seen from the graphical calculus of such categories.

Definition 6.11. A $\mathbf{STC}^\star$ is called *even* if $\Theta(X) = \mathrm{id}_X$ for all objects X.

It is easy to see that the $\mathbf{STC}^\star$ of finite dimensional Hilbert spaces as well as the $\mathbf{STC}^\star$ of unitary representations of a compact group are even. This shows that a category which is of this form must be even too.

Proposition 6.12. *If a $\boldsymbol{STC^\star}$ $\boldsymbol{C}$ admits a $\star$-preserving symmetric fiber functor E, then it is even.*

Proof. We have $\Theta(E(X)) = E(\Theta(X))$ [HM06, prop. A.45], and since $\mathcal{H}\mathbf{ilb}$ is even we have $\Theta(E(X)) = \mathrm{id}_{E(X)} = E(\mathrm{id}_X)$. But E is faithful, so we must have $\Theta(X) = \mathrm{id}_X$. □

Interestingly, being even is the only prerequisite for an $\mathbf{STC}^\star$ to be a representation category.

Theorem 6.13 (Tannaka-Krein duality, [HM06, thm. B.11])**.** *Every even $\boldsymbol{STC^\star}$ admits a $\star$-preserving symmetric fiber functor.*

Combining this with theorem 6.7, we see that every even $\mathbf{STC}^\star$ is of the form $\mathbf{Rep}_{\mathcal{H}}(G)$ for some compact group G.

[11] Here $\mathbf{Rep}_{\mathcal{H}}(G)$ denotes the category of finite dimensional unitary representations of G.

6.3 Doplicher-Roberts reconstruction theorem

There is a fairly rigorous algebraic approach to quantum field theory given by the so-called Haag-Kastler axioms. This algebraic quantum field theory is focused on the Heisenberg picture of quantum mechanics, viewing the observables of the theory as the most fundamental quantities. An algebraic quantum field theory over the Minkowski spacetime $\mathcal{M}$ is given by a *net of local observables*, which is a copresheaf

$$\mathfrak{A} : \mathcal{O}(\mathcal{M}) \to \mathbf{C^{\star}Alg}_{\mathbb{C}}$$

from the poset category of open double cones in the Minkowski space to the category of C*-algebras, such that all the morphisms $\mathfrak{A}(O_1 \subset O_2)$ are isometries. One interprets the C*-algebra $\mathfrak{A}(O)$ as the algebra of observables measurable in the region O. Since the category of C*-algebras is complete, we can form the directed limit

$$\mathfrak{A} := \varinjlim_{O \in \mathcal{O}(\mathcal{M})} \mathfrak{A}(O)$$

which, by abuse of notation, I will also denote by $\mathfrak{A}$. This is called the quasilocal algebra. The concept of relativity is introduced in form of causality of measurements. In particular, the net of local observables has to satisfy *microcausality*:

Definition 6.14. Two open double cones O_1 and O_2 are *space-like separated* if $(x-y)^2 < 0$ for all $x \in O_1$ and $y \in O_2$. Here x^2 is understood as $\langle x, x \rangle_{\mathcal{M}}$ with the bilinear form of the Minkowski space. A copresheaf $\mathfrak{A} : \mathcal{O}(\mathcal{M}) \to \mathbf{C^{\star}Alg}_{\mathbb{C}}$ satisfies *microcausality* if for all space-like separated regions O_1 and O_2 the commutator

$$[\mathfrak{A}(O_1), \mathfrak{A}(O_2)]$$

vanishes.

The observable algebras $\mathfrak{A}(O)$ used here are abstract C*-algebras in the sense that they are not given as bounded operators on some Hilbert space. Therefore, to interpret elements of these algebras as measurements of physical states we need to investigate representations of the abstract observables. By construction, it suffices to examine representations of the quasilocal algebra $\mathfrak{A}$; one can then restrict to a certain region to obtain representations of the local observables $\mathfrak{A}(O)$. In their two papers entitled "Local observables and particle statistics", S. Doplicher, R. Haag and J.E. Roberts discussed so called *superselection rules* which are used to decide which of all the representations of the quasilocal algebra are physically relevant. One can summarize these rules in the following slogan.

Definition 6.15 (DHR superselection rule)**.** The expectation values of all observables must approach the vacuum expectation values for measurements at infinitely far spacetime points.

I will formulate this criterion more precisely. Assume that we are given a fixed *vacuum representation* (H_0, π_0) of the quasilocal algebra [12].

Definition 6.16. A *DHR-representation* is a representation $\pi : \mathfrak{A} \to U(H)$ such that $\pi|_{\mathfrak{A}(O')}$ and $\pi_0|_{\mathfrak{A}(O')}$ are unitarily equivalent for all open double cones O. Here O' denotes the causal complement of the set O, which is the set theoretical complement of the causal cone of O in Minkowski space. The DHR-representations from a category $\mathbf{DHR}(\mathfrak{A})$ with morphisms being the usual bounded intertwining operators.

[12] By "representation" here I will always mean continuous $\star$-homomorphism from the C*-algebra into the algebra of unitary operators on a Hilbert space.

An unitary equivalence class of irreducible DHR-representations is called *superselection sector* of the net $\mathfrak{A}$. One can interpret non-equivalent DHR-representations as different excitations of the vacuum state, or more heuristically, the irreducible equivalence classes as particle types of the quantum field theory. In this case, the conjugate representation of a DHR-representation corresponding to a certain particle type corresponds to the antiparticle type.

Now suppose that (H, π) is some fixed representation of $\mathfrak{A}$. Then for any endomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$, the composite $\pi \circ \rho$ also is a representation. In this manner, if we fix a vacuum representation (H_0, π_0), every endomorphism of the quasilocal algebra corresponds to a representation.

Definition 6.17. Let ρ be an endomorphism of $\mathfrak{A}$ and O an open double cone. We call ρ *localized in the region* O if $\rho(A) = A$ for all $A \in \mathfrak{A}(O')$. We say that ρ is localized if there exists an open double cone O such that ρ is localized in O.

Definition 6.18. Let $\rho : \mathfrak{A} \to \mathfrak{A}$ be localized in O. We call ρ *transportable* if for any other double cone O_1 there exists an endomorphism ρ_1 of $\mathfrak{A}$ and a unitary operator $U \in \mathfrak{A}$ such that

$$U\rho(A) = \rho_1(A)U.$$

Denote by $\Delta(O)$ the transportable endomorphisms of $\mathfrak{A}$ localized in O, and by Δ the class of transportable endomorphisms of $\mathfrak{A}$. In view of the correspondence of endomorphisms and representations discussed above, the canonical choice of arrows between transportable endomorphisms is given by

$$\mathrm{Hom}_\Delta(\rho, \sigma) = \{T \in \mathfrak{A} : T\rho(A) = \sigma(A)T \text{ for all } A \in \mathfrak{A}\}$$

with composition given by the multiplication in $\mathfrak{A}$. The unit $\mathbb{1} \in \mathfrak{A}$ serves as the identity arrow of each transportable endomorphism. With these definitions, it is clear that Δ is a $\star$-category with the $\star$-operation inherited from $\mathfrak{A}$. Further, one can define a tensor product on Δ by setting $\rho \otimes \sigma = \rho \circ \sigma$ for objects and $T \otimes S = S\rho(T)$ for morphisms $S : \rho \to \rho'$ and $T : \sigma \to \sigma'$ with tensor unit given by the identity $\mathrm{id}_{\mathfrak{A}}$.

A lengthy calculation shows that this tensor product indeed makes Δ into a tensor $\star$-category (see e.g. [HM06, section 8]). The microcausality condition on the net of local observables makes it possible to define a braiding, since one can simple "transport" two transportable endomorphisms to a spacetime region where they commute. For spacetime dimensions of three or higher, this braiding is indeed a symmetry [HM06, prop. 8.50].

Interestingly, the endomorphisms of $\mathfrak{A}$ that are localized and transportable correspond exactly to the representations which satisfy the DHR-criterion.

Proposition 6.19. *There is a functor $F : \Delta \to \boldsymbol{DHR}(\mathfrak{A})$ of $\star$-categories such that $F(\rho) = \pi_0 \circ \rho$ for objects ρ of Δ and $F(s) = \pi_0(s)$ for homomorphisms $s : \rho \to \sigma$. This functor is an equivalence of categories.*

The proof is a straight forward calculation and can be found in [HM06, 8.57]. The importance of this result lies in the fact that the category Δ possesses more structure than the original DHR-category in the sense that we have an monoidal product. This will enable us to use Tannaka-style reconstruction theorems on this category.

Usually, in a more classical approach to quantum field theory, one is given not only the algebra of observables, but the full algebra of potentially unobservable fields acting on a Hilbert space of states along with a group of gauge symmetries acting on the fields such that the observables are exactly the fields invariant under the gauge symmetry. But in the beginning of this section I said that in the algebraic approach the observables are the fundamental quantities, and not the algebra of all fields. This entails that we should be able to recover the full data of the theory, i.e. the field algebra and the gauge group, from the net of local observables.

Let Δ_f be the full subcategory of Δ of objects that have a conjugate. Then by construction Δ_f is a $\mathbf{STC}^\star$, but it is not even because the twist is given by ± 1 depending on weather the sector is fermionic or bosonic. Therefore, we cannot use the Tannaka-Krein duality theorem from the previous section here. Instead, in the article "A new duality theory for compact groups" [DR89], Doplicher and Roberts proved a generalization of this result: The *Doplicher-Roberts reconstruction theorem.*

The Doplicher-Roberts reconstruction theorem. Recall from remark 5.12 that one can view an affine group scheme as an affine supergroup scheme, and then investigate the category of super representations of this group scheme. In the same manner, given a compact topological group G and an element k in the center of G of order two, one can consider the category $\mathbf{Rep}_{\mathcal{H}}(G,k)$ of representations of G on finite dimensional Hilbert spaces with the symmetry given by the $\mathbb{Z}_2$-grading induced by k: If (H,π) is a representation of G, then $\pi(k)$ yields a decomposition of H into the plus and minus one eigenspaces, making H a *super Hilbert space.* The symmetry is then given as in the case of super vector spaces (see equation (4)).

Lemma 6.20. *The category $\mathbf{Rep}_{\mathcal{H}}(G,k)$ is a $\mathbf{STC}^\star$, and the twist $\Theta((H,\pi))$ is given by $\pi(k)$.*

Proof. The first claim is clear from the properties of the category of super vector spaces and the category of representations of a compact group. In the category of finite dimensional Hilbert spaces, the conjugate of a Hilbert space H is given by the Hilbert space dual $H^\vee$, and if $\{e_i\}_{i\in I}$ is a basis of H with dual basis $\{f^i\}_{i\in I}$, the corresponding morphisms are given by $r=\sum_{i\in I} f^i\otimes e_i$ and $\bar{r}=\sum_{i\in I} e_i\otimes f^i$. One easily verifies that these are intertwiners for the trivial representation on the Hilbert space $\mathbb{C}$ and the tensor product of the representations π on H and $\overline{\pi}$ on $H^\vee$, where $\overline{\pi}$ is the complex conjugate representation of π. Therefore $(H^\vee,\overline{\pi})$ is a conjugate object of (H,π), and it even is a standard conjugate. Now let $\{e_i\}_{i\in I}$ be the eigenvector basis of $\pi(k)$, where the first l basis elements correspond to the $(+1)$ eigenspace and the others to the (-1) eigenspace. The adjoint map $r^\star$ is given by the evaluation $H^\vee\otimes_{\mathbb{C}} H\to\mathbb{C}$, $\varphi\otimes v\mapsto\varphi(v)$, and from definition 6.8 the twist is then defined by

$$\begin{aligned}\Theta((H,\pi)) =&(H\simeq\mathbb{C}\otimes_{\mathbb{C}} H\xrightarrow{r\otimes\mathrm{id}_H} H^\vee\otimes H\otimes H\xrightarrow{\mathrm{id}_{H^\vee}\otimes\tau_{H,H}} H^\vee\otimes H\otimes H\xrightarrow{r^\star\otimes\mathrm{id}_H}\mathbb{C}\otimes H\simeq H)\\ &e_j\mapsto 1\otimes e_j\mapsto\sum_{i\in I} f^i\otimes e_i\otimes e_j\mapsto\sum_{i\in I} s(i,j)f^i\otimes e_j\otimes e_i\mapsto s(j,j)e_j,\end{aligned}$$

where $s(i,j)$ is (-1) if i and j are greater than l, and $(+1)$ else. Since $s(j,j)e_j$ is equal to $\pi(k)e_j$ by construction, this proves the claim. □

Proposition 6.21. *Let G be a compact group. Then the unitary monoidal natural endomorphisms of the identity functor on $\mathbf{Rep}_{\mathcal{H}}(G)$ form an abelian group isomorphic to the center of G.*

Proof. Let k be an element of the center and (H,π) an irreducible representation of G. Then $\pi(k):H\to H$ is an intertwiner for the representation π, and therefore by Schur's lemma we must have $\pi(k)=\omega_{(H,\pi)}\mathrm{id}_H$ for some scalar $\omega_{(H,\pi)}$. Define a family of morphisms $\alpha_{(H,\pi)}=\omega_{(H,\pi)}\mathrm{id}_{(H,\pi)}:(H,\pi)\to(H,\pi)$ for all irreducible representations and extend this to a natural endomorphism of the identity functor (this is possible since the representation theory of compact groups is completely reducible, see [FH04]). This clearly is a unitary monoidal natural isomorphism. Conversely, let α be such a unitary monoidal natural endomorphism of the identity functor. Then if K denotes the forgetful functor from $\mathbf{Rep}_{\mathcal{H}}(G)$ to the category of Hilbert spaces, $K(\alpha)$ is an unitary monoidal natural transformation of K, and by Tannaka-Krein duality there is an element

$g \in G$ such that $K(\alpha_{(H,\pi)}) = \pi(g)$ for all representations (H,π) of G. Since K sends morphisms to themselves, this entails $\alpha_{(H,\pi)} = \pi(g)$, and therefore $\pi(g)$ must be an intertwiner for π. Since this is true for all representations, g must be an element of the center. □

The Doplicher-Roberts theorem shows that the generalization from the category $\mathbf{Rep}_{\mathcal{H}}(G)$ to $\mathbf{Rep}_{\mathcal{H}}(G,k)$ already suffices to describe all $\mathbf{STC}^\star$'s, not just even ones.

Theorem 6.22 (Doplicher-Roberts reconstruction theorem). *Let $\boldsymbol{C}$ be a $\boldsymbol{STC}^\star$. Then there exists a compact group G together with an element k of order two in the center of G and an equivalence $F : \boldsymbol{C} \to \boldsymbol{Rep}_{\mathcal{H}}(G,k)$ of symmetric tensor $\star$-categories. In particular, if $K : \boldsymbol{Rep}_{\mathcal{H}}(G,k) \to \boldsymbol{s\mathcal{H}ilb}$ is the forgetful functor to the category of super Hilbert spaces, the composite $E := K \circ F : \boldsymbol{C} \to \boldsymbol{s\mathcal{H}ilb}$ is a super fiber functor.*

Sketch. Define a "bosonization" $\tilde{\mathbf{C}}$ of $\mathbf{C}$ as follows: As tensor $\star$-category, let $\tilde{\mathbf{C}} = \mathbf{C}$, and define a new braiding by

$$\tilde{\tau}_{X,Y} = (-1)^{(1-\Theta(X))(1-\Theta(Y))/4} \tau_{X,Y},$$

where Θ is the twist in $\mathbf{C}$. One verifies that $\tilde{\mathbf{C}}$ is indeed a $\mathbf{STC}^\star$ and that it is even as such. Then by theorem 6.13 there is a compact group G such that $\tilde{\mathbf{C}} \simeq \mathbf{Rep}_{\mathcal{H}}(G)$. Applying proposition 6.21 to this category, the twist Θ in $\mathbf{C}$ defines a unitary monoidal endomorphism of the identity functor and therefore corresponds to an element k in the center of G such that $\Theta((H,\pi)) = \pi(k)$ for all (H,π) in $\mathbf{Rep}_{\mathcal{H}}(G)$, and since $\Theta((H,\pi))^2 = \mathrm{id}_H$ we get $k^2 = e$. One then goes on to show that $\mathbf{C} \simeq \mathbf{Rep}_{\mathcal{H}}(G,k)$, which essentially follows since $\mathbf{C}$ is related to $\tilde{\mathbf{C}}$ in the same way that $\mathrm{Rep}_{\mathcal{H}}(G,k)$ is related to $\mathbf{Rep}_{\mathcal{H}}(G)$. □

A full proof of this can be found in [HM06, thm. B.18]. Using this theorem on the category Δ_f, one can now recover the gauge group of the theory along with the distinction between bosonic and fermionic fields, proving that indeed the net of local observables is the more fundamental quantity.

Remark 6.23. It is worth noting that these representations (H,π) in $\mathbf{Rep}_{\mathcal{H}}(G,k)$ come equipped with a decomposition $H = H_B \oplus H_F$ into a bosonic and fermionic subspace given by the eigenspaces of $\pi(k)$, but since the group G itself is still purely even there are no symmetries in G mixing bosonic and fermionic states. In particular, $\pi(k)$ is equivariant since k is in the center of G, and therefore every element of G must preserve the eigenspaces of this operator. In this sense, the terminology "supergroup" for a tuple (G,k) prevalent in physics literature is very misleading, as in a supersymmetric theory the symmetry group must indeed relate fermionic and bosonic states.

6.4 Deligne's theorem and supersymmetry

In comparison to the Doplicher-Roberts reconstruction theorem, the theorem of Deligne actually concerns itself with a generalization of groups to supergroups. In this section I will show how this relaxation appears naturally in particle physics, and how this in turn strengthens the belief that our particle model admits a supersymmetry.

Wigner's classification of fundamental particles. The Poincaré group is the isometry group of the Minkowski spacetime, and as such encompasses translations as an abelian subgroup. The infinitesimal generators of these translations in the directions of the standard basis elements of $\mathcal{M}$ form a quintuple $P := (E, P_1, P_2, P_3)$, which in physics is called the momentum operator. Recall now that in special relativity the rest mass of a particle is related to its energy and momentum by the energy-momentum relation $E^2 = \vec{p}^2 + m^2$. If the Poincaré group acts on a Hilbert space H and $v \in H$ is an eigenvector of the operator $P^2 := E^2 - P_1^2 - P_2^2 - P_3^2$ with eigenvalue μ^2, then we have

$$P^2 v = E^2 v - P_1^2 v - P_2^2 v - P_3^2 v = \mu^2 v,$$

and therefore one interprets μ as the rest mass of the particle in state v. One can show that the operator P^2 is a Casimir element of the Poincaré group, and therefore the rest mass μ together with the eigenvalues of the other Casimir element, which are interpreted as the spin, labels the irreducible unitary representations of the Poincaré group.

This connection between the defining features of fundamental particles and classification of irreducible representations of the Poincaré group lead to *Wigner's classification of elementary particles*, which states that in a quantum field theory, the elementary particle species are in one-to-one correspondence with the irreducible representations of the symmetry group of the theory. By the Coleman-Mandula theorem mentioned in section 6.1, in case that the geometric symmetries are given by the Poincaré group, the full symmetry group is a direct product of the Poincaré group and the internal symmetry group, so that the fundamental particles are classified by the irreducible representations of these two groups separately.

The category of particle species. Note that the category of unitary representations of the symmetry, of which the irreducible representations form the simple objects, naturally admits the structure of a tensor category. Even more general, it may be suggested that every sensible collection of particle species forms a tensor category: Suppose we are given some abstract collection of objects that we know to be the particle species of our theory, together with all the possible interactions between these particle species. This information can be assembled into a category **Ptcl** whose objects are the particle species and whose arrows between two objects P and Q are all the possible ways an instance of the species P can undergo an interaction that results in an instance of species Q. To make this into an abelian category, we define the hom-sets between two particle species to be not just those arrows but their $\mathbb{C}$-linear span. The simple objects of this category will correspond to the fundamental particle species of the theory. In a physical theory we expect there to be a possibility to make two particle species P and Q into a new particle species $P \otimes Q$ whose instances are compound systems of instances of P and Q. Note that the tensor product of two fundamental particle species does not have to be fundamental anymore, just as the tensor product of two irreducible representations usually is not irreducible. For example when given a Fock space of an elementary particle, two-particle states are given by the tensor product of the one-particle Hilbert space with itself. We also expect there to exist some type of vacuum I that serves as a tensor unit for this monoidal product of forming compound systems. A braiding on this tensor product is given by exchanging the instances of the particle species forming the tensor product, and this braiding clearly is a symmetry, since we expect the system to be invariant under exchanging two particles twice. Finally, we assume that every sensible theory admits the notion of an anti-particle species $P^\vee$ for every particle species P. This entails that we have a notion of a pair production morphisms $I \to P \otimes P^\vee$ and of a particle-antiparticle annihilation $P^\vee \otimes P \to I$ which make the object $P^\vee$ a dual object of P.

In this exposition I followed the article [Sch16]. If one believes that all these elements, compound systems, a vacuum and anti-particles, are necessary parts of a physical theory of elementary particles, then the category **Ptcl** will have the structure of a complex tensor category. If in addition one believes that there should only be finitely many fundamental particles, as is customary in most theories, this tensor category will be of subexponential growth (see [Del02, cor. 0.7]), and we can apply Deligne's theorem on tensor categories 5.20. As a result we see that there is an affine supergroup G and an element $p \in G(\mathbb{C})$ such that

$$\mathbf{Ptcl} \simeq \mathbf{Rep}(G, p).$$

Then, again employing Wigner's classification of fundamental particles, we see that the affine supergroup G must be the spacetime symmetry group of our theory. Now this does not mean that every spacetime symmetry group of a sensible physical theory must be supersymmetric; this is evidently false since the standard model exhibits all of the above described features and is not supersymmetric. But it means that the most general context in which Wigner's classification and all the basic properties we expect from particle species can be true is a supersymmetric theory.

If one believes that the standard model is not yet a complete description of particle physics, then this gives a good reason why one should look at supersymmetry as an extension of this theory.

Discussion of the result. As stated in the introduction, this interaction of Deligne's theorem with supersymmetry in quantum field theory was the original motivation for this thesis. It may thus be asked how convincing this argumentation really is, and it which way it might be improved. For a start, I have not specified very well what I mean by "particle species", but rather left it as an abstract notion. Now in particle physics, one in fact uses Wigner's classification, or to be more precise the so called quantum numbers, to label the particle species of the standard model, so a physicist might ask what could be meant by particle species if not "irreducible representation of the symmetry group". The same issue arises with the tensor product of two particle species: What exactly is an instance of this tensor product? One cannot simply take one instance of each particle species, since we want to allow interaction between particles, and thus a system of two different particles can take on many different particle contents. The tensor product in the DHR-category avoids this issue by using the local nature of interactions, so there might be hope that there is a well-defined tensor product employing methods from scattering theory. I have not been able to define such an explicit monoidal structure, and further I have not been able to define the action of this monoidal product on the morphisms of the particle species category. The connection between dual objects and anti-particles there can also be found some deviation from the standard model. For example, the annihilation of an electron-positron pair does not leave behind the vacuum, but rather two photons carrying the energy and momentum of the electron and positron. Very similarly, the electron-positron pair production emerges from an already existing photon rather than the vacuum. Pair production and annihilation from and to the vacuum only handles so called virtual particles, which cannot be observed.

At last, if one works trough all the issues with the category of particle species, there remains the issue that while Deligne's theorem talks about group objects (or supergroup objects for that matter) in the category of affine schemes and their representations, in physics one usually requires a topological group and its unitary representations. Also, Wigner did use all the unitary representations of the geometric symmetry group, not only the finite dimensional ones. For the Poincaré group, this problem may be bypassed by using the representation theory for semi-direct products, which only requires knowledge of some particular finite dimensional representations, but this does not have to work in a general case.

I hope that moving forward there will be solutions found to all these inconsistencies.

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