Hochschild Cohomology and Higher Centers

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Plan

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1. Higher centers
2. Hochschild cohomology
3. The geometric case
4. Motivation and WIP

Higher Centers

Centers as universal objects

Throughout the talk, let k be a field of characteristic 0.

$A \in \mathsf{Alg}_k$

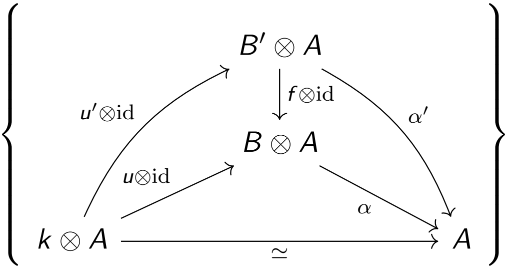
Internal endomorphism object of A = object representing the functor

$$B \mapsto \mathrm{Hom}_{\mathsf{Alg}_k}(B \otimes A, A)$$

usually does not exist.

$\rightsquigarrow$ Such an object *would be* an algebra object in Alg_k via composition, hence a commutative k-algebra. It would canonically act on A by evaluation.

But: There exists a **universal commutative k-algebra acting on A**, i.e. a final object of



$\rightsquigarrow$ This universal object is the **center** of A

$$Z(A) \otimes A \xrightarrow{\text{mult.}} A$$

Derived centers

Definition (Lurie)

Let $\mathcal{D}$ be a monoidal ∞-category, and $A \in \mathcal{D}$. A **center** of A is a final object

$$\mathfrak{Z}(A) \in \mathrm{LMod}(\mathcal{D}) \times_{\mathcal{D}} \{A\}$$

Have forgetful functor

$$\mathrm{LMod}(\mathcal{D}) \times_{\mathcal{D}} \{A\} \to \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D})$$

$\rightsquigarrow$ Identify the center of A with an object

$$\mathfrak{Z}(A) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D})$$

Operadic centers

We are interested in $\mathcal{D} = \mathsf{Alg}_{\mathcal{O}}(\mathcal{C})$ for ∞-operad $\mathcal{O}$ and SM ∞-category $\mathcal{C}$.

$\rightsquigarrow \mathfrak{Z}(A) \in \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Alg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathsf{Alg}_{\mathbb{E}_1 \otimes \mathcal{O}}(\mathcal{C})$

Our example: $\mathcal{C} = \mathsf{Vect}_k$, $\mathcal{O} = \mathbb{E}_1 = \mathcal{A}ssoc$, $\mathcal{D} = \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Vect}_k) = \mathsf{Alg}_k$

$\rightsquigarrow \mathfrak{Z}(A) = Z(A) \in \mathsf{Alg}_{\mathbb{E}_1 \otimes \mathbb{E}_1}(\mathsf{Vect}_k) \simeq \mathsf{Alg}_{\mathbb{E}_\infty}(\mathsf{Vect}_k)$

is a commutative k-algebra.

Dunn additivity

In general:

Theorem (Lurie)

Let $\mathcal{C}$ be a SM ∞-category. Then there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})).$$

Corollary

If $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, then

$$\mathfrak{Z}(A) \in \mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C}).$$

Slogan: The center of an $\mathbb{E}_k$-algebra is the universal $\mathbb{E}_{k+1}$-algebra acting on it.

Hochschild Cohomology

The Hochschild complex

Classically: Hochschild cohomology = "derived center"

$$C^*(A, A) \simeq \mathbb{R}\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A) \simeq \mathrm{Hom}_k(A^{\otimes *}, A),$$
$$\mathrm{HH}^0(A, A) \cong Z(A)$$

Hochschild cohomology admits algebraic structure:

- Cup product corresponding to the Yoneda product (of degree 0)
- Gerstenhaber bracket (of degree -1)

$\rightsquigarrow$ $\mathrm{HH}^*(A, A)$ is a **Gerstenhaber algebra**

New definition of Hochschild cochains

Definition

Let $\mathcal{C}$ be a (nice enough) k-linear SM ∞-category, and let $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. The Hochschild complex of A is the center

$$\mathfrak{Z}(A) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}).$$

This definition has a "built-in" solution to

Deligne's conjecture on Hochschild cochains

The Hochschild cochain complex of an associative k-algebra is an algebra over the chains on little 2-disks operad, such that the induced Gerstenhaber structure on cohomology recovers the cup product and classical Gerstenhaber bracket.

Gerstenhaber structure on Hochschild cohomology

$A \in \mathsf{Alg}_k \hookrightarrow \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{D}(k))$

$\mathfrak{Z}(A) \in \mathsf{Alg}_{\mathbb{E}_2}(\mathcal{D}(k)) \xleftarrow[\simeq]{\text{Rectification}} \mathsf{Alg}_{C_*(\mathbb{E}_2)}(\mathsf{Ch}(k))^c[W^{-1}]$

$\mathrm{H}_*(\mathfrak{Z}(A)) \in \mathsf{Alg}_{\underbrace{H_*(\mathbb{E}_2)}_{\simeq \mathbf{Ger}}}(\mathsf{Ch}(k))$

$\rightsquigarrow$ Does this recover the classical cup product and Gerstenhaber bracket?

Comparison theorem

Theorem (F.)

Let $A \in \mathrm{Alg}_k \hookrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}(k))$.

1. *The underlying object and module action of* $\mathfrak{Z}(A)$ *are equivalent to* $C^*(A, A) = \mathrm{Hom}_k(A^{\otimes *}, A)$ *with the evaluation map*

$$C^*(A, A) \otimes A \to A.$$

2. *The induced* ***Ger****-algebra structure in cohomology of the center agrees with the classical cup product and Gerstenhaber bracket on Hochschild cohomology.*

Comparison theorem

Corollary

The center $\mathbb{E}_2$-structure actually solves Deligne's Conjecture.

Proof sketch

1. is straight forward using

Theorem (Lurie)

If it exists, the endormorphism object

$$\mathrm{End}_{\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})}(A) \in \mathcal{C}$$

of A as an $\mathbb{E}_1$-module over itself is the underlying object of the center of A.

+ some techincal results identifying $\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{D}(k)) \simeq N_{\mathrm{dg}}(\mathrm{Ch}(A \otimes A^{\mathrm{op}})^{\circ})$

2. is the interesting part. We need to understand the $\mathbb{E}_2$-structure of the center.

$\rightsquigarrow$ Have $\mathfrak{Z}(A) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}(k)) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}(k)))$, so we can break up the problem into two steps:

1. Find the $\mathbb{E}_1 \otimes \mathbb{E}_1$-algebra structure on $\mathfrak{Z}(A)$
2. Find out how to compute the cup product and Gerstenhaber bracket of the $\mathbb{E}_2$-algebra corresponding to an $\mathbb{E}_1 \otimes \mathbb{E}_1$-algebra

Corollary (to Prop. 5.3.1.29 HA, F.)

Assume that the morphism object

$$\mathrm{End}_{\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})}(A) \in \mathcal{C}$$

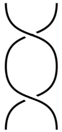
exists. Then the "inner" multiplication of the center is given by the convolution product, the "outer" multiplication is given by the composition product, and there is a contractible choice of fillings of the compatibility square

$$\begin{array}{ccc} \mathfrak{Z}(A)^{\otimes 4} & \xrightarrow{\circ\otimes\circ} & \mathfrak{Z}(A)^{\otimes 2} \\ {\scriptstyle (\star\otimes\star)(id\otimes\tau\otimes id)}\downarrow & & \downarrow{\scriptstyle \star} \\ \mathfrak{Z}(A)^{\otimes 2} & \xrightarrow[\circ]{} & \mathfrak{Z}(A) \end{array}$$

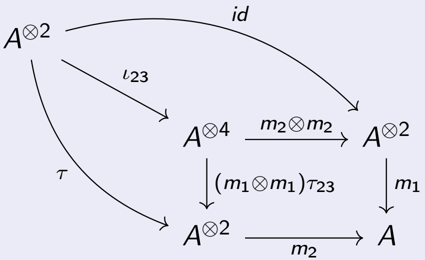
in $\mathcal{C} \times_{\mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})} \mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})/A$.

Theorem (F.)

Let $A \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{E}_1}(\mathcal{C})$*. The homotopy class of the double twist operation*

$\gamma =$ 

in the corresponding $\mathbb{E}_2$*-algebra is a composition of the four "Eckmann-Hilton 2-simplices".*



+ check:

- $\star$ and $\circ$ correspond to the classical cup product
- the classical circle product yields a filler for the compatibility square

The Geometric Case

The geometric case

Let X be an algebraic variety $/k$.

Direct generalization of Hochschild cochain complex (Swan, Gestenhaber-Schack, Grothendieck-Loday):

$$C^*(X) := \mathbb{R}\mathcal{H}om_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

Problem: Does not come equipped with a Gerstenhaber bracket (not even in cohomology)

The smooth case

Let X be a **smooth** algebraic variety $/k$.

Definition/Proposition (Kontsevich)

There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{D}^*_{\text{poly}}(X)$, the sheaf of ***polydifferential operators****, with*

$$\mathcal{D}^*_{\text{poly}}(X)(\text{Spec}A) \xrightarrow{\sim} C^*(A, A)$$

given by maps $A^{\otimes n} \to A$ that are differential operators in each variable. This is a sheaf of Gerstenhaber algebras in the category of complexes of sheaves of k-vector spaces.

Set $C^*(X) := \mathcal{D}^*_{\text{poly}}(X)$. Then

$$\text{HH}^*(X) := \mathbb{H}^*(X, \mathcal{D}^*_{\text{poly}}(X))$$

inherits the structure of a Gerstenhaber algebra.

The new definition

Let X be a quasi-compact separated scheme $/k$. Let $\mathcal{C} = \mathrm{dgSh}(X)$ be the SM ∞-category of dg sheaves on X. Then

$$\mathcal{O}_X \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{dgSh}(X)) \xrightarrow{\text{forget}} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{dgSh}(X))$$

Definition

The Hochschild cochain complex of X is given by the center

$$C^*(X) := \mathfrak{Z}(\mathcal{O}_X) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{dgSh}(X)).$$

In particular: This equips

$$\mathrm{HH}^*(X) = \mathbb{H}^*(X, \mathfrak{Z}(\mathcal{O}_X))$$

with a Gerstenhaber algebra structure, **even in the singular case**.

$\rightsquigarrow$ We want to argue that this is a "good" definition.

Local properties

Theorem (F.)

Let $U = \mathrm{Spec}(A) \subseteq X$ *be an affine open. Then*

$$\mathbb{R}\Gamma_U(\mathfrak{Z}(\mathcal{O}_X)) \simeq \mathfrak{Z}(A)$$

in $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}(k))$.

This is the analogue of the fact that $\mathcal{D}^*_{\mathrm{poly}}(X)$ affine locally recovers the classical Hochschild complex of the algebra.

This is noteworthy, since centers are in general **not** functorial.

Global comparison theorem

Theorem (F.)

Let X be a smooth quasi-compact variety $/k$.

1. $\mathcal{D}^*_{\mathrm{poly}}(X) \simeq \mathfrak{Z}(\mathcal{O}_X) \in \mathrm{dgSh}(X)$.
2. *The induced* ***Ger****-algebra structure on* $\mathbb{H}^*(X, \mathfrak{Z}(\mathcal{O}_X))$ *agrees with the classical one on* $\mathbb{H}^*(X, \mathcal{D}^*_{\mathrm{poly}}(X))$.

In particular, the center $\mathbb{E}_2$-algebra structure is the "correct" $\mathbb{E}_2$-algebra structure on Hochschild cochains.

Motivation and WIP

Motivation and WIP

Let X be a smooth variety $/k$.

Generalized Kontsevich formality theorem:

Theorem (Calaque-Van den Bergh)

$$\mathbb{H}^*(X, \mathcal{T}^*_{\text{poly}}(X)) \xrightarrow{\text{HKR}\circ \text{Td}(X)^{1/2}\wedge -} \mathbb{H}^*(X, \mathcal{D}^*_{\text{poly}}(X))$$

is an isomorphism of Gerstenhaber algebras.

This is a geometric version of the **Duflo theorem** in Lie algebra theory.

In terms of centers

My work: $\mathcal{D}^*_{\text{poly}}(X) \simeq \mathfrak{Z}_{\mathbb{E}_1}(\mathcal{O}_X)$

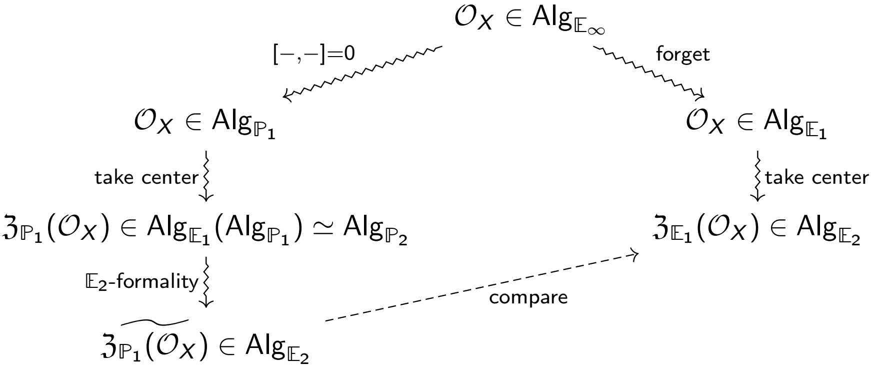
Work in progress:

Conjecture (Safronov)

The sheaf of polyvector fields is the Poisson center of $\mathcal{O}_X$ (with the trivial Poisson bracket):

$$\mathcal{T}^*_{\text{poly}}(X) \simeq \mathfrak{Z}_{\mathbb{P}_1}(\mathcal{O}_X)$$

$\rightsquigarrow$ Use this to reformulate the Formality Theorem in terms of centers



Questions

- There is no corresponding formality between modules over $\mathbb{E}_1$-algebras in the category of $\mathbb{E}_1$-algebras and modules over $\mathbb{E}_1$-algebras in the category of $\mathbb{P}_1$-algebras. But a comparison map between the Poisson and $\mathbb{E}_1$-centers would correspond to a quantization of the canonical action

$$\mathfrak{Z}_{\mathbb{P}_1}(\mathcal{O}_X) \otimes \mathcal{O}_X \to \mathcal{O}_X$$

- How does the Todd class come into play?
- Where does such a comparison map live? (Have an $\mathbb{A}_\infty$ no-go theorem for the Lie case)

The Grothendieck-Teichmüller group

An $\mathbb{E}_2$-formality map requires a choice of a **Drinfeld Associator**. The collection of these form a torsor of the **Grothendieck-Teichmüller group**.

Definition (Fresse)

The (pro-unipotent) Grothendieck-Teichmüller group is given by

$$\mathrm{GT}(\mathbb{Q}) := \pi_0 \mathrm{Aut}^{\mathrm{h}}_{\mathcal{O}p}(\widehat{\mathbb{E}}_2^{\mathbb{Q}}).$$

This group is closely related to the absolute Galois group of the rationals, and to this day remains mysterious.

The DRW action

Theorem (Dolgushev-Rogers-Willwacher)

Let X be a smooth variety over k. We have a group action

$$\mathrm{GT}(\mathbb{Q}) \circlearrowright \left\{ \begin{array}{c} \text{Ger-isomorphisms} \\ \mathbb{H}^*(X, \mathcal{T}^*_{\text{poly}}(X)) \xrightarrow{\simeq} \mathbb{H}^*(X, \mathcal{D}^*_{\text{poly}}(X)) \\ \text{correcting HKR} \end{array} \right\}$$

which is non-trivial and non-torsor for certain choices of X.

$\rightsquigarrow$ This was done using T. Willwacher's correspondence between the GT Lie algebra and the zeroth cohomology of the Kontsevich graph complex.

In terms of centers

Since GT($\mathbb{Q}$) acts on $\mathbb{E}_2$-formality maps, we expect it to also act on comparisons of Poisson and $\mathbb{E}_1$-centers.

In addition, by definition it acts on algebras over rationalization of the $\mathbb{E}_2$-operad. In particular, we expect it to act on $\mathbb{E}_1$-centers in $\mathbb{Q}$-linear categories.

Question: Can we recover the DRW action of the Grothendieck-Teichmüller group in the center picture?